You may need the following facts:

- Suppose that $y_1(t)$, which is not everywhere zero, is a solution of
  
  $y'' + p(t)y' + q(t) = 0$.

  To find a second solution, one can set $y_2(t) = v(t)y_1(t)$. Then $v(t)$ satisfies
  
  $y_1v'' + (2y'_1 + py_1)v' = 0$.

- If $p, q, g$ are continuous functions over some open interval $I$, and
  
  $y'' + p(t)y' + q(t)y = 0$

  has general solution $c_1y_1(t) + c_2y_2(t)$, then non-homogeneous equation
  
  $y'' + p(t)y' + q(t)y = g(t)$

  has a particular solution
  
  $Y(t) = y_1(t) \int \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt$. 
1. Consider the initial value problem

\[ y'' - y' - 2y = 0, \quad y(0) = 0, y'(0) = \alpha. \]

   a. Find its solution.

b. Determine how the behavior of \( y(t) \) as \( t \to \infty \) depends on \( \alpha \).
2. Consider the second order differential equation

\[ y'' - 6y' - 7y = 4e^{3t} + 5e^{-t}. \]

a. Find the complementary solution.

b. Use the method of undetermined coefficients to find a particular solution and formulate the general solution.
3. Consider the equation

\[ t^2 y'' + 7ty' + 5y = t, \quad t > 0, \quad y_1(t) = t^{-1}. \]

a. Verify that \( y_1(t) = t^{-1} \) is a solution.

b. Use the method of reduction of orders (variation of parameters) to find a second solution \( y_2(t) \).

c. Calculate \( W(y_1, y_2) \) to verify that \( y_1, y_2 \) are linearly independent.
d. Use results in (b) and the method of variation of parameters to find a particular solution.

e. Write the general solution of the differential equation.
4. A mass of 3 kg is attached to a spring vertically with the spring constant $k = 15 \text{ N/m}$. Suppose that there is no damping force. At $t = 0$, the mass is further stretched down 0.2 meter from the equilibrium and is then released.

   a. Formulate an initial value problem that describes the motion of the mass.

   b. Find the displacement of the mass from the equilibrium at any time $t > 0$.

   c. Suppose that in addition, there is an external force $F(t) = 4\cos(\omega t)$. What value of $\omega$ would produce resonance?
d. Under the external force in (c) and the same initial conditions, find the displacement of the mass from the equilibrium at any time $t > 0$. 
5. A mass of 5 kg is attached to a spring vertically with the spring constant $k = 5 \text{ N/m}$. Suppose there is a damping force with damping coefficient $\gamma \text{ N} \cdot \text{s/m}$. At $t = 0$, the mass is stretched further down 0.1 meter from equilibrium and is then released.

   a. Find the condition of $\gamma$ so that this vibration is critically-damped, underdamped, or overdamped, respectively.

   b. In the case where the vibration is critically damped, find the displacement of the mass from the equilibrium at any time $t > 0$.

   c. Determine whether the mass will pass through the equilibrium.
d. Suppose that in addition, there is an external force $F(t) = 20 \sin t$. Find the steady state solution under such an external force.

e. Find the amplitude of the steady state solution in (d).