Least-Squares

Distance from a Point to a Plane

Vectors $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.

The distance from point $P$ to plane $W$ is $||\vec{y} - \hat{\vec{y}}||$.

**Note:** Vectors $\vec{u}_1$ and $\vec{u}_2$ must be orthogonal!
- Plane $W$ is spanned by vectors $\vec{u}_1$ and $\vec{u}_2$.
- Vector $\tilde{\vec{y}}$ is not in plane $W$.
- Vector $\hat{\vec{y}}$ is the orthogonal projection of $\tilde{\vec{y}}$ onto plane $W$.
- Vector $\vec{y} - \hat{\vec{y}}$ is in $W^\perp$. 
Least-Squares

Problem: \( A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \) and \( \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} \).

Solve for \( \vec{x} \) in \( A\vec{x} = \vec{b} \).

Solution: 

\[
\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 11 \end{bmatrix} \rightarrow \text{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

So, there’s no solution!

Or is there?

Is there some value that’s “close enough” that we can sort of call a solution?

Is that allowed? Why can’t mathematicians just accept that there’s no solution?
The reason the system in our example has no solution is because $\vec{b}$ is not in the span of the column vectors of matrix $A$.

Consider this picture:

Plane $W$ represents $\text{Col}(A)$, spanned by the column vectors of $A$, which are $\vec{x}_1$ and $\vec{x}_2$.

If the system had a solution, vector $\vec{b}$ would also be in the plane. But it isn’t, so …

What’s the closest we can get to the plane?

Although there is no $\vec{x}$ for which $A\vec{x} = \vec{b}$, there is an $\hat{\vec{x}}$ which minimizes the distance $\|\vec{b} - A\hat{\vec{x}}\|$ between the plane and vector $\vec{b}$.

$\hat{\vec{x}}$ is known as the least-squares solution.

To find $\hat{\vec{x}}$, we solve the following equation:

$$A^T A \hat{\vec{x}} = A^T \vec{b}$$

The set of resulting equations are known as the normal equations.

The distance $\|\vec{b} - A\hat{\vec{x}}\|$ is known as the least-squares error.
Example 1

Find the least-squares solution $\hat{x}$ of the system $A\hat{x} = \vec{b}$, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.

What is the SSE (Sum-of-Squares Error)? What is the least-squares error?
Example 2

Find the least-squares solution $\hat{x}$ of the system $A\hat{x} = \bar{b}$, where $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

What is the SSE (Sum-of-Squares Error)? What is the least-squares error?
**Example 3**
Using least squares, fit a quadratic function of the form $f(t) = c_0 + c_1 t + c_2 t^2$ to the data points $(0, 4), (1, -3), (2, -4), (3, -19)$. 