Orthogonality

The Inner Product (Dot Product)

The inner product is the multiplication of two vectors of the same dimension that yields a scalar:

\[ \mathbf{y} \cdot \mathbf{u} = y_1 u_1 + y_2 u_2 + \cdots + y_n u_n \]

\[ \mathbf{y} \cdot \mathbf{u} = \|\mathbf{y}\| \|\mathbf{u}\| \cos \theta \]

\[ \|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \]

is known as the “norm” or “magnitude” of the vector.

\( \theta \) represents the angle between the vectors.

Example 1

Find the inner product of the following pairs of vectors, and use the result to figure out the angle between the vectors.

(a) \( \mathbf{y} = [1 \ 0] \) and \( \mathbf{u} = [1 \ 1] \)

(b) \( \mathbf{y} = [1 \ 0] \) and \( \mathbf{u} = [0 \ 1] \)
**Orthogonal Vectors**

If the dot product of two vectors is zero, we know that the vectors are perpendicular to each other.

We call perpendicular vectors **orthogonal**.

**Unit Vectors**

When the “length” or “magnitude” or “norm” of a vector equals 1, we call it a **unit vector**.

To find the magnitude of a vector, we use this formula:

The norm of \( \vec{v} \) is \( \| \vec{v} \| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \) (this is the Pythagorean Theorem! 😊)

To make a vector into a unit vector, we divide it by its norm (we normalize it):

\[ \frac{\vec{v}}{\| \vec{v} \|} \]

**Example 2**

Find a unit vector \( \vec{u} \) in the same direction as \( \vec{v} = \begin{bmatrix} 5 \\ 12 \end{bmatrix} \).
Orthonormal Vectors
Unit vectors that are orthogonal are called orthonormal vectors.

Coordinates in the orthogonal world
Let \( \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \} \) be an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \).
For each \( \vec{y} \) in \( W \), the weights (coordinates) in the linear combination
\( \vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_p \vec{u}_p \)
are given by
\[
c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}, \quad (j = 1, \ldots, p)
\]

NOTE: When the basis is orthonormal, this formula simplifies to \( c_j = \vec{y} \cdot \vec{u}_j \).

Example 3
Given \( \vec{v} = \begin{bmatrix} -6 \\ -1 \\ -2 \end{bmatrix} \), find the coordinates for \( \vec{v} \) in the subspace \( W \) spanned by the orthogonal vectors

\( \vec{u}_1 = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} \).
Orthogonal Projections

The orthogonal projection of a vector onto another vector is like a “shadow” vector that the top vector casts onto the other one:

\[ \hat{y} = \text{proj}_{\bar{u}} \bar{y} \]

Notice that the distance between point P and vector \( \bar{u} \) is the magnitude of the vector \( \bar{y} \) minus the orthogonal projection.

\[ \hat{y} = \text{proj}_{\bar{u}} \bar{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u} \]

The distance from point \( P \) to vector \( \bar{u} \) is \( ||\bar{y} - \hat{y}|| \).

Example 4

Find the distance from point \( P \) (3, 4) to vector \( \bar{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \).
**Example 5**

Let \( \vec{y} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix} \), \( \vec{u} = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix} \).

Write \( \vec{y} \) as the sum of two orthogonal vectors, \( \vec{x}_1 \) in Span(\( \vec{u} \)) and \( \vec{x}_2 \) orthogonal to \( \vec{u} \).
Example 6

Let $L$ be the line given by the span of $\begin{bmatrix} -4 \\ 7 \\ -3 \end{bmatrix}$ in $\mathbb{R}^3$.

Find a basis for the orthogonal complement $L^\perp$ of $L$. 