Systems of Linear ODEs: Homogeneous with Repeated Eigenvalues

Generalized Eigenvectors
Recall that a defective matrix has fewer linearly independent eigenvectors than eigenvalues. In order to form a complete basis, we can find extra eigenvectors called generalized eigenvectors.

To find a generalized eigenvector \( \tilde{v}_2 \), solve the equation

\[
(A - \lambda I) \tilde{v}_2 = \tilde{v}_1
\]

Example 1
Find the generalized eigenvector of

\[
A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}
\]
Solution of Systems with Repeated Eigenvalues

Consider the linear system $\mathbf{x}' = A \mathbf{x}$, where $A$ is a $2 \times 2$ matrix.

If the eigenvalues of $A$ are real and equal, the general solution to the linear system will be:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 \left( \mathbf{v}_1 t + \mathbf{v}_2 \right) e^{\lambda t}$$

where $\mathbf{v}_1$ is the regular eigenvector and $\mathbf{v}_2$ is the generalized eigenvector.

Example 2

Suppose that the matrix $A$ has repeated eigenvalue with the following eigenvector and generalized eigenvector:

$\lambda = 3$ with eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ and generalized eigenvector $\mathbf{w} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$.

Write the solution to the linear system $\mathbf{r}' = A \mathbf{r}$ in the following forms.

A. In eigenvalue/eigenvector form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right) e^{3t}$$

B. In fundamental matrix form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -2 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

C. As two equations: (write "c1" and "c2" for $c_1$ and $c_2$)

$$x(t) =$$

$$y(t) =$$
Example 3

Find the general solution of the linear system of differential equations.

\[
\begin{align*}
    x'_1 &= x_1 - x_2 \\
    x'_2 &= x_1 + 3x_2
\end{align*}
\]
Summary
A linear system of homogeneous ODEs can be converted to a matrix equation.

The system

\[
\begin{align*}
    x_1' &= ax_1 + bx_2 \\
    x_2' &= cx_1 + dx_2
\end{align*}
\]

is equivalent to

\[
\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

which we usually write as

\[
\mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}
\]

The solution to this system will depend on the eigenvalues of the coefficient matrix.

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<th>Solution</th>
<th>Notation</th>
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<td>Real ( \lambda_1 \neq \lambda_2 )</td>
<td>( \mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} )</td>
<td>( \mathbf{v}_1 ) and ( \mathbf{v}_2 ) are eigenvectors</td>
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<tr>
<td>Real ( \lambda_1 = \lambda_2 )</td>
<td>( \mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} )</td>
<td>( \mathbf{v}_1 ) is an eigenvector ( \mathbf{v}_2 ) is a generalized eigenvector</td>
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<tr>
<td>Complex ( \lambda = \alpha \pm \beta i )</td>
<td>( \mathbf{x}(t) = e^{\alpha t} (c_1 \mathbf{v}<em>{real} + c_2 \mathbf{v}</em>{imaginary}) )</td>
<td>( \mathbf{v}<em>{real} ) and ( \mathbf{v}</em>{imaginary} ) are the vectors we get after distributing and simplifying the complex eigenvector in the general solution: ((\mathbf{w} + i\mathbf{z})e^{(\alpha + \beta i)t})</td>
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