Vector Spaces and Subspaces

A set of vectors* is a vector space if the following requirements are met:

1. The zero vector is in the set.
2. Adding two vectors in the set results in a vector that’s also in the set.
3. Multiplying a vector in the set by any scalar results in a vector that’s also in the set.

*Keep in mind that a “vector” can be any mathematical object: polynomials, matrices, etc.

Example of a set that is a vector space: \( W = \left\{ \begin{bmatrix} a \\ b - c \\ c + a \end{bmatrix}, a, b, c \in \mathbb{R} \right\} \)

Checking the requirements:

1. When \( a, b, c \) are zero, we can create the zero vector, \( \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).
2. If we add two vectors that have the form \( \begin{bmatrix} a \\ b - c \\ c + a \end{bmatrix} \), we always get a vector with the same pattern: \( \begin{bmatrix} a_1 \\ b_1 - c_1 \\ c_1 + a_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 - c_2 \\ c_2 + a_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ (b_1 + b_2) - (c_1 + c_2) \\ (c_1 + c_2) + (a_1 + a_2) \end{bmatrix} \).
3. If we multiply a vector of this form \( \begin{bmatrix} a \\ b - c \\ c + a \end{bmatrix} \) by a scalar, we still get a vector with that pattern: \( k \begin{bmatrix} a \\ b - c \\ c + a \end{bmatrix} = \begin{bmatrix} ka \\ kb - kc \\ kc + ka \end{bmatrix} \).

Example of a set that is not a vector space: \( W = \left\{ \begin{bmatrix} 5 \\ b - c \\ c + a \end{bmatrix}, a, b, c \in \mathbb{R} \right\} \)

Checking the requirements:

1. No matter what \( a, b, c \) are, we can never create the zero vector, \( \vec{0} \), because the first entry is always 5.
2. If we add two vectors that have the form \( \begin{bmatrix} 5 \\ b - c \\ c + a \end{bmatrix} \), we always get a vector with 10 in the first entry, not 5.
3. If we multiply a vector of this form \( \begin{bmatrix} 5 \\ b - c \\ c + a \end{bmatrix} \) by a scalar, the top entry will be 5 times the scalar, which will not be 5 unless the scalar is 1.
Example 1

Show that $W = \left\{ \begin{bmatrix} a + 2b \\ c \\ 3b - c \end{bmatrix}, \ a, \ b, \ c \in \mathbb{R} \right\}$ is a vector space and find a suitable basis for it.
Example 2

Show that \( W = \left\{ \begin{bmatrix} a - b \\ 2c \\ b + c \end{bmatrix}, a + b = 3, c \in \mathbb{R} \right\} \) is not a vector space.
A subset of vectors of a vector space is a **subspace** if the following requirements are met:

1. The zero vector of the subset is the same as the zero vector of the vector space.
2. Adding two vectors in the subset results in a vector that’s also in the subset.
3. Multiplying a vector in the subset by any scalar results in a vector that’s also in the subset.

**Example of a subset that is a subspace:**

\[ W = \left\{ \begin{bmatrix} a \\ 3a \\ -2a \end{bmatrix}, a \in \mathbb{R} \right\} \text{ is a subspace of } \mathbb{R}^3 \]

Checking the requirements:

1. When \( a = 0 \), the zero vector of \( W \) is \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \), which is the same as the zero vector of \( \mathbb{R}^3 \).
2. If we add two vectors that have the form \( \begin{bmatrix} a \\ 3a \\ -2a \end{bmatrix} \), we always get a vector with the same pattern.
3. If we multiply a vector of this form \( \begin{bmatrix} a \\ 3a \\ -2a \end{bmatrix} \) by a scalar, we still get a vector with that pattern.

**Example of a subset that is NOT a subspace:**

\[ W = \left\{ \begin{bmatrix} a \\ 3a \\ -2a \end{bmatrix}, a > 0 \right\} \text{ is a subset of } \mathbb{R}^3 \]

Checking the requirements:

1. Since \( a \) is restricted to be a positive number, we can never create the zero vector of \( \mathbb{R}^3 \).
2. If we add two vectors that have the form \( \begin{bmatrix} a \\ 3a \\ -2a \end{bmatrix} \), we always get a vector with the same patterns.
3. If we multiply a vector of this form \( \begin{bmatrix} a \\ 3a \\ -2a \end{bmatrix} \) by a negative scalar, the top entry will be negative, but \( a \) cannot be negative.
Example 3

$\mathbb{P}_2$ is the vector space of all polynomials of degree \textit{at most} two.

Are the following subsets of $\mathbb{P}_2$ also subspaces of $\mathbb{P}_2$?

(a) $\{ p(x) \mid p(-x) = p(x) \text{ for all } x \}$

(b) $\{ p(x) \mid \int_0^8 p(x) \, dx = 0 \}$

(c) $\{ p(x) \mid p(5) = 3 \}$
(d) \{ p(x) \mid p'(x) \text{ is constant} \}

(e) \{ p(x) \mid p'(0) = p(3) \}

(f) \{ p(x) \mid p'(x) + 6p(x) + 5 = 0 \}
Example 4
Is the subset of all $2 \times 2$ matrices whose trace is zero a subspace of $\mathbb{M}_{2\times 2}$?

If so, find a basis for it.