Systems of Linear ODEs and Eigenvectors Review

When we have higher-order differential equations, it is useful to rewrite them as a *system* of first-order differential equations. This is useful because we can represent the system with matrix notation and use our linear algebra skills to solve it.

**Example 1**

Consider the following second-order IVP:

\[
\begin{cases}
    u'' - 5u' - 3u = -8 \sin(3t) \\
    u(1) = 2.5 \\
    u'(1) = 4
\end{cases}
\]

Let \( v = u' \) and rewrite the differential equation as an equivalent set of first-order equations.

\[
\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u' \\ v' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} u(1) \\ v(1) \end{bmatrix} = \begin{bmatrix} \_ \\ \_ \end{bmatrix}
\]
Example 2
Write the given third order linear equation as an equivalent system of first order equations with initial values.

\[
\begin{aligned}
&y''' - y' \cos t = -2 \sin(t) \\
y(1) = 3 \\
y'(1) = 2 \\
y''(1) = 0
\end{aligned}
\]

Use \( x_1 = y, x_2 = y', x_3 = y'' \).

\[
\begin{aligned}
\tilde{x}' &= \begin{bmatrix} \end{bmatrix} \tilde{x} + \begin{bmatrix} \end{bmatrix} \\
\tilde{x}( ) &= \begin{bmatrix} \end{bmatrix}
\end{aligned}
\]
**Eigenvalues Review**

An **eigenvector** of an $n \times n$ matrix is a nonzero vector that remains on its own span when it is multiplied by that matrix. This phenomenon is summarized by the equation:

$$A\vec{v} = \lambda \vec{v}$$

When we multiply matrix $A$ by its corresponding eigenvector $\vec{v}$, we get a scalar multiple of the eigenvector. That scalar multiple is called the **eigenvalue**, and it is represented by the Greek letter $\lambda$ (lambda).

Eigenvalues can be real or complex, and so can eigenvectors.

**Equations**

To solve for the **eigenvalues**, we use this equation:

$$\det(A - \lambda I) = 0$$

To solve for the **eigenvectors**, we use this equation:

$$(A - \lambda I)\vec{v} = \vec{0}$$

**Useful Properties**

1. If a matrix is *originally* upper or lower triangular, or diagonal, the eigenvalues are found on its diagonal entries.

2. The trace of a matrix equals the sum of the eigenvalues: $\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

3. The determinant of a matrix equals the product of its eigenvalues: $\det(A) = \lambda_1\lambda_2 \cdots \lambda_n$
Example 3
Find the eigenvalues and eigenvectors of matrix $A$.

$$A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$$
Diagonalization Review

An \( n \times n \) matrix is **diagonalizable** if and only if it has \( n \) linearly independent eigenvectors.

If so, the matrix can be factored using this formula:

\[
A = PD P^{-1}
\]

Matrix \( D \) contains the eigenvalues of matrix \( A \), and \( P \) contains the corresponding eigenvectors.

\[
D = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}, \quad P = [ \vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_n ]
\]

Things to keep in mind:

1. All these matrices are square and have the same dimension, \( n \times n \).
2. The eigenvalues and eigenvectors are real (not complex).

Raising a diagonalizable matrix to a power

If you want to raise a diagonalizable matrix to some power \( k \), use the following formula:

\[
A^k = PD^k P^{-1}
\]
Example 4

Find a suitable diagonalization of matrix $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$. 