Reduction of Order and Undetermined Coefficients

Reduction of Order

Let’s say we have a second-order differential equation $y'' + p(t)y' + q(t)y = 0$. Imagine that we already know one solution $y_1(t)$.

How can we find a second solution? One way is with Abel’s Theorem. Here’s another way.

Assume the second solution is a nonconstant multiple of the first:

$$y_2 = v(t)y_1(t)$$

Now take the first and second derivatives:

$$y'_2 = v'y_1 + vy'_1$$
$$y''_2 = v''y_1 + 2v'y'_1 + vy'''_1$$

Substitute $y_2$ into original differential equation:

$$y''_2 + p(t)y'_2 + q(t)y_2 = 0$$
$$v''y_1 + 2v'y'_1 + vy'''_1 + p(v'y_1 + vy'_1) + qvy_1 = 0$$

Rearrange as a second-order differential equation of $v$:

$$y_1v'' + (2y'_1 + py_1)v' + (y''_1 + p(t)y'_1 + q(t)y_1)v = 0$$

Because $y_1$ is a solution, this part $y''_1 + p(t)y'_1 + q(t)y_1$ must equal zero.

And our differential equation reduces to:

$$y_1v'' + (2y'_1 + py_1)v' = 0$$

This equation is really first-order for $v'$!

We can easily see that with a simple substitution. Let $w = v'$:

$$y_1w' + (2y'_1 + py_1)w = 0$$

Once you solve for $w$, you can integrate it to find $v$. Finally, $y_2 = vy_1$. 

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**Example 1**

Use the method of reduction of order to find a second solution of the given differential equation.

\[ t^2 y'' + 2ty' - 2y = 0, \quad t > 0, \quad y_1(t) = t \]
**Undetermined Coefficients**

Suppose we have a second-order ODE with constant coefficients, but now the right side is no longer zero.

\[ ax'' + bx' + cx = f(t), \quad a, b, c \in \mathbb{R} \]

The solution to the nonhomogeneous differential equation is made up of two parts:

\[ x(t) = x_c(t) + x_p(t) \]

Complementary solution: \( x_c(t) \) solves the homogeneous differential equation

Particular solution: \( x_p(t) \) solves the nonhomogeneous differential equation

We can find \( x_p(t) \) with the method of **undetermined coefficients**.

This method works when \( f(t) \) is:
- \( e^t \)
- Polynomial \( t^n + t^{n-1} + t^{n-2} + \cdots \)
- \( \sin t \) or \( \cos t \)
- Any combination of the above functions

How the method works:

1. Solve the homogeneous case to determine \( x_c(t) \).
2. Determine \( x_p(t) \) by “guessing.”
   - The guess will depend on \( f(t) \) and the roots of the characteristic polynomial.

The phrase “undetermined coefficients” refers to the coefficients that we have to figure out in our guess for \( x_p(t) \).
### How to guess

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>Roots of Characteristic Equation</th>
<th>Particular Solution Guess</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{at}$</td>
<td>$\lambda_1, \lambda_2 \neq \alpha$</td>
<td>$x_p = Ae^{at}$</td>
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<tr>
<td></td>
<td>$\lambda_1 \neq \alpha, \lambda_2 = \alpha$</td>
<td>$x_p = At e^{at}$</td>
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<td></td>
<td>$\lambda_1 = \lambda_2 = \alpha$</td>
<td>$x_p = At^2 e^{at}$</td>
</tr>
<tr>
<td>Polynomial of degree $n$: $t^n$</td>
<td>$\lambda_1, \lambda_2 \neq 0$</td>
<td>$x_p = At^n + Bt^{n-1} + Ct^{n-2} + \ldots + D$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 = 0$ or $\lambda_2 = 0$</td>
<td>$x_p = t(At^n + Bt^{n-1} + Ct^{n-2} + \ldots + D)$</td>
</tr>
<tr>
<td>$\sin(\beta t)$ or $\cos(\beta t)$</td>
<td>$\lambda \neq \alpha \pm i\beta$</td>
<td>$x_p = A \cos(\beta t) + B \sin(\beta t)$</td>
</tr>
<tr>
<td></td>
<td>$\lambda = \alpha \pm i\beta$</td>
<td>$x_p = t(A \cos(\beta t) + B \sin(\beta t))$</td>
</tr>
</tbody>
</table>
Example 2
Find the general solution of the given differential equation using the method of undetermined coefficients.

\[ x'' - 2x' - 15x = e^{4t} \]
Example 3

Find the general solution of the given differential equation using the method of undetermined coefficients.

\[ x'' - 2x' - 15x = 2 \sin(3t) \]
Example 4
Find the general solution of the given differential equation using the method of undetermined coefficients.

\[ x'' - 2x' - 15x = 4t^2 \]