Repeated Roots

Last time we used Abel’s Theorem to develop a general solution to a differential equation when its characteristic equation $a\lambda^2 + b\lambda + c = 0$ has real, repeated roots $\lambda_1 = \lambda_2$.

The general solution to the second-order differential equation when $\lambda_1 = \lambda_2$ is

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

**Example 1**

Find the general solution to the differential equation $x'' - 8x' + 16x = 0$. 
**Complex Roots**

What happens when the roots are complex? That is, when $\lambda = \alpha \pm \beta i$.

Let’s review Euler’s Formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Since the solutions have the form $e^{\lambda t}$, we can use $\lambda = \alpha + \beta i$ with Euler’s formula:

$$e^{\lambda t} = e^{(\alpha + \beta i)t} = e^{\alpha t} e^{i\beta t}$$

$$= e^{\alpha t} (\cos (\beta t) + i \sin (\beta t))$$

But we don’t want the $i$ ... We would like “real-valued functions.”

Since sine and cosine are linearly independent functions, they form a fundamental set.

Therefore, the general solution to the differential equation will be:

$$x(t) = e^{\alpha t} (c_1 \cos (\beta t) + c_2 \sin (\beta t))$$
Example 1

Find the general solution of the given differential equation and describe its long-term behavior.

\[ x'' + 4x = 0 \]
Example 2
Find the solution of the given initial-value problem and describe its long-term behavior.

\[ x'' + 4x' + 5x = 0, \quad x(0) = 1, \quad x'(0) = 0 \]
Example 3

Find the solution of the given initial-value problem and describe its long-term behavior.

\[ x'' - 2x' + 5x = 0, \quad x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 2 \]
Summary of Solutions to 2\textsuperscript{nd} Order ODEs with Constant Coefficients

Differential equation: \( ax'' + bx' + cx = 0, \quad a, b, c \in \mathbb{R} \)

Characteristic equation: \( a\lambda^2 + b\lambda + c = 0 \)

<table>
<thead>
<tr>
<th>Roots of characteristic equation</th>
<th>Solution of differential equation</th>
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<tbody>
<tr>
<td>real, ( \lambda_1 \neq \lambda_2 )</td>
<td>( x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} )</td>
</tr>
<tr>
<td>real, ( \lambda_1 = \lambda_2 )</td>
<td>( x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} )</td>
</tr>
<tr>
<td>complex, ( \lambda = \alpha \pm \beta i )</td>
<td>( x(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)) )</td>
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</tbody>
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Complex Roots and Oscillating Solutions

<table>
<thead>
<tr>
<th>( \lambda = \alpha \pm \beta i )</th>
<th>Oscillation Type</th>
<th>Differential Equation Solution Graph</th>
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<tbody>
<tr>
<td>( \alpha = 0 )</td>
<td>Steady</td>
<td><img src="image1.png" alt="Graph" /></td>
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<tr>
<td>( \alpha &gt; 0 )</td>
<td>Growing</td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
<tr>
<td>( \alpha &lt; 0 )</td>
<td>Decaying</td>
<td><img src="image3.png" alt="Graph" /></td>
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