Integration Review

**Example 1 – Power rule, sine and cosine, $e^x$**

\[
\int (3x - \sin x + e^{5x}) \, dx = \frac{3}{2} x^2 + \cos x + \frac{e^{5x}}{5} + C
\]

You can always check the answer of an integral by taking its derivative:

\[
\frac{d}{dx} \left( \frac{3}{2} x^2 + \cos x + \frac{e^{5x}}{5} + C \right) = 3x - \sin x + e^{5x}
\]

Notice that the result is exactly what you started out with in the integral.

**Example 2 – Substitution**

\[
\int \frac{1}{3x - 2} \, dx
\]

Let $u = 3x - 2$

Then $du = 3\, dx$ ... or $dx = \frac{1}{3} \, du$.

Therefore:

\[
\int \frac{1}{3x - 2} \, dx = \int \frac{1}{u} \left( \frac{1}{3} \, du \right) = \frac{1}{3} \int \frac{1}{u} \, du = \frac{1}{3} \ln |u| + C
\]

Remember to turn the answer back to the original variable:

\[
\frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |3x - 2| + C
\]

\[
\int \frac{1}{3x - 2} \, dx = \frac{1}{3} \ln |3x - 2| + C
\]
Example 3 – Substitution

\[ \int 5x e^{x^2} \, dx \]

Let \( u = x^2 \)
Then \( du = 2x \, dx \)

Therefore:

\[ \int 5x e^{x^2} \, dx = \frac{5}{2} \int e^u \, du = \frac{5}{2} e^u + C \]

Converting back to original variable:

\[ \int 5x e^{x^2} \, dx = \frac{5}{2} e^{x^2} + C \]

Checking the result:

\[ \frac{d}{dx} \left( \frac{5}{2} e^{x^2} + C \right) = 5x e^{x^2} \]
Integration by Parts

Integration by parts comes from the product rule of differentiation. To derive the formula, start with a product of two functions, such as \( uv \), and take the derivative with respect to \( x \).

\[
\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}
\]

Now multiply the entire result by \( dx \).

\[
dx\left(\frac{d}{dx}(uv)\right) = \frac{du}{dx}v\,dx + u\frac{dv}{dx}\,dx
\]

Simplify:

\[
\frac{d}{dx}(uv)\,dx = v\,du + u\,dv
\]

Integrate everything:

\[
\int \frac{d}{dx}(uv)\,dx = \int v\,du + \int u\,dv
\]

\[
uv = \int v\,du + \int u\,dv
\]

Rearrange into familiar formula:

\[
\int u\,dv = uv - \int v\,du
\]

The idea with this formula is to pick for \( u \) the function that becomes “simpler” when you differentiate (\( x^2 \) becomes \( 2x \), for example). Pick the \( dv \) to be the part that is differentiable. Keep in mind that the resulting \( v\,du \) must be integrable.
Example 4 – Integration by Parts

\[ \int x \sin x \, dx \]

Formula: \( \int u \, dv = uv - \int v \, du \)

Which function becomes simpler when you take its derivative? Is it \( x \) or \( \sin x \)? It’s \( x \)!

Let \( u = x \) and \( dv = \sin x \, dx \)
Then \( du = dx \) and \( v = -\cos x \)

Therefore:

\[ \int x \sin x \, dx = -x \cos x - \int -\cos x \, dx \]

\[
\int x \sin x \, dx = -x \cos x + \sin x + C
\]

How would you know if you picked \( u \) and \( dv \) incorrectly?
Let \( u = \sin x \) and \( dv = x \, dx \)
Then \( du = \cos x \, dx \) and \( v = \frac{x^2}{2} \)

Therefore:

\[ \int x \sin x \, dx = \frac{x^2}{2} \sin x - \int \frac{x^2}{2} \cos x \, dx \]

Notice that the integral on the right-hand side is even worse than the original one!
That’s how you would know. ☺
Partial Fractions

The idea with partial fractions is to break up a fraction into fractions whose denominators are the factors of the denominator of the original fraction.

For example, we could say:

\[
\frac{5}{6} = \frac{1}{2} + \frac{1}{3}
\]

**Example 5 – Partial Fractions**

\[
\int \frac{3}{x^2 + 4x - 21} \, dx
\]

Rewrite the fraction with a factored denominator:

\[
\frac{3}{(x + 7)(x - 3)}
\]

Make an equation with fractions whose denominators are the factors:

\[
\frac{3}{(x + 7)(x - 3)} = \frac{A}{x + 7} + \frac{B}{x - 3}
\]

Multiply by the common denominator:

\[
(x + 7)(x - 3) \left( \frac{3}{(x + 7)(x - 3)} = \frac{A}{x + 7} + \frac{B}{x - 3} \right)
\]

\[
3 = A(x - 3) + B(x + 7)
\]

Think of the 3 as 3 + 0x and rearrange the equation into an x term and a number term:

\[
3 + 0x = (-3A + 7B) + (A + B)x
\]

Therefore, \(A + B\) must be 0, and \(A = -B\).

The \(-3A + 7B\) must be equal to 3: \(-3A + 7B = 3\)

Substitute \(A = -B\): \(3B + 7B = 3\)

Therefore, \(B = \frac{3}{10}\) and \(A = -\frac{3}{10}\).

Finally, substitute the results into the original integral and integrate:

\[
\int \frac{3}{x^2 + 4x - 21} \, dx = \int \left[ \left( -\frac{3}{10} \right) \frac{1}{x + 7} + \left( \frac{3}{10} \right) \frac{1}{x - 3} \right] \, dx
\]

\[
= -\frac{3}{10} \ln|x + 7| + \frac{3}{10} \ln|x - 3| + C
\]