Transformation

A transformation (or mapping), denoted \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a rule that assigns each vector \( \vec{x} \) in \( \mathbb{R}^n \) (the domain) an output vector \( T(\vec{x}) \) in \( \mathbb{R}^m \) (the codomain). \( T(\vec{x}) \) is often called the image of \( \vec{x} \) and the set of all possible images is called the range of \( T \).

A transformation is a linear transformation if for \( \vec{u}, \vec{v} \in \mathbb{R}^n \) and \( c \) any scalar,
\[
T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})
\]
and
\[
T(c\vec{u}) = cT(\vec{u}).
\]
Notice that \( \vec{u} + \vec{v} \) and \( c\vec{u} \) occur in \( \mathbb{R}^n \) while \( T(\vec{u}) + T(\vec{v}) \) and \( cT(\vec{u}) \) occur in \( \mathbb{R}^m \).

Example: Is \( T(\begin{bmatrix} x \\ y \end{bmatrix}) = xy \) a linear transformation (L.T.)?
Let \( \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \), then \( c\vec{u} = \begin{bmatrix} cx \\ cy \end{bmatrix} \)
\[
T(c\vec{u}) = (cx)(cy) = c^2xy
\]
and \( cT(\vec{u}) = c(xy) = cxy \) So \( \text{NO} \)

Students try: Is \( T(\begin{bmatrix} x \\ y \end{bmatrix}) = x + 2y + a \), \( a \in \mathbb{R} \), a L.T.? 
Answer \( \text{NO} \)
The only value of \( a \) allowed is \( a = 0 \).

Notice: \( T(c\vec{u}) = cT(\vec{u}) \Rightarrow T(\vec{0}) = \vec{0} \)
A very important result is that every L.T. \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is equivalent to a matrix multiplication, i.e. \( T(\vec{x}) = \vec{A}\vec{x} \) where \( \vec{A} \) is an \( m \times n \) matrix called the standard matrix for the L.T. This is often denoted as: \( \vec{x} \mapsto \vec{A}\vec{x} \)

Now we see why we consider \( \mathbb{R}^n \) as the input space and \( \mathbb{R}^m \) as the output space — so that \( \vec{A} \) is of dimension \( m \times n \). This matrix is easy to compute since it is:

\[
\vec{A} = \begin{bmatrix}
T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n)
\end{bmatrix}
\]

i.e., the \( j \)th column of \( \vec{A} \) is just the transformation applied to the \( j \)th standard basis vector \( \vec{e}_j \). Remember \( \vec{e}_j \in \mathbb{R}^n \) consists of a 1 at the \( j \)th location and all other entries are 0.

Example: \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) defined by the rule

\[
T(\vec{x}) = T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, -x_1 + 4x_2)
\]

Note, I have written vectors horizontally to save space. But remember, \( T(\vec{e}_j) \) is still the \( j \)th column of \( \vec{A} \).

\[
\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ so that means the inputs are } x_1 = 1, x_2 = 0, x_3 = 0
\]

Thus \( T(\vec{e}_1) = (2 \cdot 1 - 0 + 0, -1 + 4 \cdot 0) = (2, -1) \)
This becomes the first column of \( \vec{A} \).
Continuing, for \( \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \), \( T(\vec{e}_2) = (-1, 1) \)
and for \( \vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \), \( T(\vec{e}_3) = (1, 0) \)

Thus \( A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & 0 \end{bmatrix} \)

Check \( Ax = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ -x_1 + 4x_2 + 0x_3 \end{bmatrix} \)

the first output element is \( 2x_1 - x_2 + x_3 \) and the second output element is \( -x_1 + 4x_2 \).

Student: The effect of a combination of a certain horizontal shear and a certain vertical expansion is illustrated:

\[ \begin{array}{c}
\begin{array}{c}
1 \\
\parallel
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
T
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
2 \\
\parallel
\end{array}
\end{array} \]

For this \( T: \mathbb{R}^2 \to \mathbb{R}^2 \), find \( A \)

Answer: \( A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \)

A mapping \( T: \mathbb{R}^n \to \mathbb{R}^m \) is said to be onto \( \mathbb{R}^m \) if each \( \vec{b} \) in \( \mathbb{R}^m \) is the image of at least one \( \vec{x} \) in \( \mathbb{R}^n \).
A linear transformation is **onto** if and only if the standard matrix $A$ has a **pivot** in every row.

A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $b$ in $\mathbb{R}^m$ is the image of at most one $x$ in $\mathbb{R}^n$.

$$
\begin{array}{c}
\begin{array}{c}
T \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T \\
\end{array}
\end{array}
$$

T is one-to-one

T is not one-to-one

A linear transformation is **one-to-one** if and only if the standard matrix $A$ has a pivot in every column.

Student: Test the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ for which we found the matrix for the properties onto and one-to-one.

Answer: $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & 0 \end{bmatrix}$

$$
\begin{bmatrix}
2 & -1 & 1 \\
-1 & 4 & 0 \\
\end{bmatrix}
\xrightarrow{RREF}
\begin{bmatrix}
1 & 0 & \frac{4}{7} \\
0 & 1 & \frac{1}{7} \\
\end{bmatrix}
$$

Pivot in every row $\Rightarrow T$ is onto

Nota: Pivot in every column $\Rightarrow T$ is not one-to-one