The Chain Rule

**THE CHAIN RULE** If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at $x$ and $F'$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

**NOTE** In using the Chain Rule we work from the outside to the inside. Formula 2 says that we differentiate the outer function $f$ (at the inner function $g(x)$) and then we multiply by the derivative of the inner function.

$$\frac{d}{dx} \left( \frac{f(g(x))}{f'(g(x))} \right) = \frac{f'(g(x))}{g'(x)}$$

In order to see this better I sometimes first understand the function from the inside to the outside and then reverse that order to take the derivative.

For example: $y = \sqrt{\sin(x^2)}$

Given an $x$, I ask myself what must I do to $x$ to evaluate the function? I first square $x$ to get $x^2$, then I take the $\sin$ of that $x^2$ to get $\sin(x^2)$ and finally I take the square root of that $\sin(x^2)$, obtaining my

$$y = \sqrt{\sin(x^2)}$$
I take the derivative in the reverse order working toward the independent variable \( x \).

So \( \frac{dy}{dx} = \frac{1}{2} (\sin(x^2))^{-\frac{1}{2}} \frac{d}{dx} (\sin(x^2)) \)

\[ = \frac{1}{2} (\sin(x^2))^{-\frac{1}{2}} \cos(x^2) \frac{d}{dx} (x^2) \]

\[ = \frac{1}{2} (\sin(x^2))^{-\frac{1}{2}} \cos(x^2) 2x \]

\[ = \frac{x \cos(x^2)}{\sqrt{\sin(x^2)}} \]

just algebraically simplified.

Some very common easy examples:

\[ y = e^{kx+b} \]

\[ \frac{dy}{dx} = e^{kx+b} \frac{d}{dx} (kx+b) = e^{kx+b} k \]

\[ = k e^{kx+b} \]

So \( \frac{d}{dx} e^{2x} = 2 \cdot e^{2x} \)

\[ y = \cos(\pi t + \frac{\pi}{3}) \]

\[ \frac{dy}{dt} = -\sin(\pi t + \frac{\pi}{3}) \frac{d}{dt} (\pi t + \frac{\pi}{3}) \]

\[ = -\pi \sin(\pi t + \frac{\pi}{3}) \]

So \( \frac{d}{dx} \sin(3x) = 3 \cos(3x) \)
Examples:

a) \( P(x) = (x^4 + 3x^2 - z)^5 \)
\[
\frac{dP}{dx} = 5 (x^4 + 3x^2 - z)^4 \frac{d}{dx} (x^4 + 3x^2 - z)
\]
\[
= 5 (x^4 + 3x^2 - z)^4 (4x^3 + 6x)
\]
\[
= 10x (x^4 + 3x^2 - z)^4 (2x^2 + 3)
\]
Last step simplification of notation only

b) \( g(t) = \frac{1}{(t^4 + 1)^3} = (t^4 + 1)^{-3} \)
\[
\frac{dg}{dt} = -3(t^4 + 1)^{-4} \frac{d}{dt} (t^4 + 1)
\]
\[
= -3(t^4 + 1)^{-4} \cdot 4t^3
\]
\[
= -12t^3
\]
\[
\frac{1}{(t^4 + 1)^4}
\]
By the quotient rule applied to the original form, we have
\[
\frac{dg}{dt} = \frac{(t^4 + 1)^3 \frac{d}{dt} (1) - 1 \frac{d}{dt} (t^4 + 1)^3}{(t^4 + 1)^6}
\]
\[
= \frac{(t^4 + 1)^3 \cdot 0 - 3(t^4 + 1)^2 \frac{d}{dt} (t^4 + 1)}{(t^4 + 1)^6}
\]
\[
= -3(t^4 + 1)^2 (4t^3)
\]
\[
\frac{1}{(t^4 + 1)^6}
\]
\[
= -12t^3
\]
\[
\frac{1}{(t^4 + 1)^4}
\]
I recommend avoiding the quotient rule if possible.
c.) \[ y = \sin(\tan 2x) \]
\[ \frac{dy}{dx} = \cos(\tan 2x) \frac{d}{dx} (\tan 2x) \]
\[ = \cos(\tan 2x) \sec^2(2x) \frac{d}{dx} (2x) \]
\[ = 2 \cos(\tan 2x) \sec^2(2x) \]

d.) \[ y = 4^x \] since \[ y = e^{\ln 4} \], we have
\[ y = (e^{\ln 4})^x = e^{\ln 4 x} \]
as equivalent.
Thus \[ \frac{dy}{dx} = e^{\ln 4 x} \frac{d}{dx} (\ln 4 x) \]
\[ = \ln 4 e^{\ln 4 x} = (\ln 4) 4^x \]

In general \[ \frac{d}{dx} (a^x) = a^x \ln a \] when \( a > 0 \)

Problems for students, find derivatives of:

a.) \[ F(x) = \sqrt{1 + 2x + x^3} \]
b.) \[ y = \cos (a^3 + x^3) \]
c.) \[ y = e^{2x} \]
d.) \[ f(x) = 2 (6\pi x + \frac{3\pi}{2})^{11} \]
e.) \[ y = e^{e^x} \]
f.) \[ y = 2 \sin(\pi x) \]