6.3 

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Cylindrical Shell Method Examples

Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 0$, and $x = 1$ about the line $x = 2$ using the method of cylindrical shells.

$h(x) = y_{\text{upper}} - y_{\text{lower}} = x^4 - 0 = x^4$

$r(x)$ = distance between slice and rotation line

$r(x) = 2 - x$ (Note, this is an $x_{\text{right}} - x_{\text{left}}$)

$$dV = 2\pi r(x) h(x) \, dx$$
$$= 2\pi (2-x) x^4 \, dx$$

Total volume = $\sum dV$

$$V = \int_0^1 2\pi (2-x) x^4 \, dx = 2\pi \left[ \frac{x^6}{6} - 2 \right]_0^1$$

$$= 2\pi \left[ \frac{2}{5} - \frac{1}{6} - (0 - 0) \right]$$

$$= 2\pi \left[ \frac{3}{30} \right] = 2\pi \left[ \frac{2}{30} \right] = \frac{7\pi}{15}$$
Cylindrical Shell

Find the volume of the solid obtained by rotating the region bounded by \( y = \frac{1}{x}, \ y = 0, \ 1 \leq x \leq 2 \) about the \( y \)-axis. (i.e. line \( x = 0 \))

See illustration taken from textbook. A representative vertical slice of the region is shown at the far left.

This forms a cylindrical shell when rotated about the \( y \)-axis. The slice has small width \( dx \) and thus the shell has small thickness \( dr \).

The small volume of the shell is

\[
dV = \text{Circumference} \times \text{Height} \times \text{Thickness}
\]

\[
dV = 2\pi r \times h \times dr
\]

\( r = x \text{ location of slice minus location of axis of revolution} = x - 0 = x \), and \( dr = dx \)

\( h = y \text{ of upper end of slice minus } y \text{ of lower end of slice} \)

\( h = \frac{1}{x} - 0 = \frac{1}{x} \)

Thus

\[
dV = 2\pi x \frac{1}{x} \ dx = 2\pi \ dx
\]

The total volume is

\[
V = \int_{1}^{2} dV = 2\pi \int_{1}^{2} \ dx = 2\pi \left[ x \right]_{x=1}^{2}
\]

\( = 2\pi (2 - 1) = 2\pi \)
Find the volume of the solid obtained by rotating the region bounded by $x = 1 + y^2$, $x = 0$, $y = 1$, and $y = 2$ about the $x$-axis, i.e., $y = 0$.

See the illustration adapted from the textbook. With the method of cylindrical shells, the slice is parallel to the line of rotation. Thus, the slice is horizontal and has small width $dy$. As this slice is rotated about the $x$-axis, cylindrical shells are produced which are horizontal. The $r(y)$, as shown, is $r(y) = y$ i.e., the slice is at $y$.

The height $h(y)$ is as shown:

\[ h(y) = x_{\text{right end}} - x_{\text{left end}} = 1 + y^2 - 0 = 1 + y^2 \]

The small volume of the shell is:

\[ dV = 2\pi r(y) h(y) dy = 2\pi y (1 + y^2) dy \]

The total volume is:

\[ V = \int dV = 2\pi \int_1^2 y (1 + y^2) dy = 2\pi \left[ \frac{y^2}{2} + \frac{y^4}{4} \right]_1^2 = 2\pi \left[ \frac{2^2}{2} + \frac{2^4}{4} - \left( \frac{1^2}{2} + \frac{1^4}{4} \right) \right] \]

\[ = 2\pi \left[ 2 + 4 - \left( \frac{1}{2} + \frac{1}{4} \right) \right] \]

\[ = 2\pi \left[ \frac{21}{4} \right] = \frac{21\pi}{2} \]