Volume of a Solid of Revolution by the Method of Discs or Washers

• Determine the given bounded region.

If the bounded region is to be rotated about a horizontal line (x-axis or another line parallel to the x-axis) then:

• Vertical slices (i.e., perpendicular to the rotation line) are made in the bounded region, with dx the small width of the slice.

• As the slice is rotated about the rotation line either a disc or washer of small thickness dx is swept out. Successive discs or washers are stacked horizontally with their centers at the rotation line.

• If one end of the slice is at the rotation line, a disc is formed, and its small volume is

\[ dV = \text{Area} \times \text{Small Thickness} = \pi R(x)^2 \, dx \]

where \( R(x) \) is the vertical distance (always positive) from the rotation line to the far end of the slice.

• If neither end of the slice is at the rotation line, a washer is formed, and its small volume is

\[ dV = \text{Area} \times \text{Small Thickness} = \pi (R(x)^2 - r(x)^2) \, dx \]

where \( R(x) \) is the vertical distance (always positive) from the rotation line to the farther end of the slice and
r(x) is the vertical distance (always positive) from the rotation line to the closer end of the slice.

- The total volume of the solid of revolution is obtained by summing all the small volumes by integration.

\[ V = \int_a^b \pi R(x)^2 \, dx \]

Thus, when discs

\[ V = \int_a^b \pi (R(x)^2 - r(x)^2) \, dx \]

and when washers

with a and b the limits of integration chosen so the slice moves throughout the bounded region.

If the bounded region is rotated about a vertical line (y-axis or a line parallel to y-axis) then:
- The slices are horizontal with small width dy.
- Rotating these slices about the rotation line again sweeps out either a disc or a washer of small thickness dy. Successive discs or washers are stacked vertically with their centers at the rotation line.
- R(y) and r(y) are now horizontal distances from the rotation line to the respective ends of the horizontal slice.

- If discs

\[ V = \int_a^b \pi R(y)^2 \, dy \]

- If washers

\[ V = \int_a^b \pi (R(y)^2 - r(y)^2) \, dy \]

Integration is with respect to y.
Volume of a Solid of Revolution
by the
Method of Cylindrical Shells
- Determine the given bounded region.

If the bounded region is to be rotated about a vertical line (y axis or another line parallel to the y axis):
- Vertical slices (i.e., parallel to the rotation line) are made in the bounded region, with dx the small width of the slice.
- As the slice is rotated about the rotation line, a cylindrical shell with small thickness dx is swept out. Its center line is the rotation line.
- A cylindrical shell results because all of the slice is the same distance, r(x), from the rotation line. Thus, r(x) is the (always positive) distance between the rotation line and the slice. The slice moves in a circle of circumference 2πr.
- The height, h(x), of the shell is the (always positive) length of the slice. Since the slice is vertical, h(x) = y(x)_{Upper end} - y(x)_{Lower end}.
- The small volume of the cylindrical shell is
dV = Area \times Small\ Thickness = Circumference \times Height \times Thickness
  = 2\pi r(x) \cdot h(x) \cdot dx
- The total volume sums dV by integrating,
V = \int_a^b 2\pi r(x) \cdot h(x) \cdot dx

The limits of integration, a and b, are chosen so the slice moves throughout the bounded region.
• Note, the solid of revolution can be thought of as made up of all the cylindrical shells nested within each other.

If the bounded region is rotated about a horizontal line (x axis or a line parallel to the x axis) then:
• The slices are horizontal with small width dy.
• When the slices are rotated about the horizontal rotation line, cylindrical shells are formed, but they are now oriented horizontally.
• The height, \( h(y) \), of the shell is now the horizontal length of the slice,
  \[
  h(y) = x(y)_{\text{Right end}} - x(y)_{\text{Left end}}
  \]
• The radius \( r(y) \) is the (always positive) distance between the rotation line and the slice. Circumference of the slice's path is \( 2\pi r \).
• The small volume of the cylindrical shell is
  \[
  dV = 2\pi r(y) \ h(y) \ dy
  \]
• The total volume sums \( dV \) by integrating
  \[
  V = \int_a^b 2\pi r(y) \ h(y) \ dy
  \]
The limits of integration, \( a \) and \( b \), are chosen so the slice moves throughout the bounded region. The integration is with respect to \( y \).