The Squeeze Theorem and Example

3 The Squeeze Theorem

If \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and

\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L
\]
then

\[
\lim_{x \to a} g(x) = L
\]

Example 11. Show that \( \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 \).

Solution. First note that we cannot use

\[
\lim_{x \to 0} x^3 \sin \frac{1}{x} = \lim_{x \to 0} x^3 \cdot \lim_{x \to 0} \sin \frac{1}{x}
\]

because \( \lim_{x \to 0} \sin(1/x) \) does not exist (see Example 4 in Section 2.2). However, since

\[-1 \leq \sin \frac{1}{x} \leq 1\]

we have, as illustrated by Figure 8,

\[-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2\]

We know that

\[
\lim_{x \to 0} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0} (-x^2) = 0
\]

Taking \( f(x) = -x^2 \), \( g(x) = x^2 \sin(1/x) \), and \( h(x) = x^2 \) in the Squeeze Theorem, we obtain

\[
\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0
\]

Matching Derivatives to Their Functions

Match the following functions

\[\text{A = } \begin{array}{c} \includegraphics[width=3cm]{A.png} \end{array}\]
\[\text{B = } \begin{array}{c} \includegraphics[width=3cm]{B.png} \end{array}\]
\[\text{C = } \begin{array}{c} \includegraphics[width=3cm]{C.png} \end{array}\]
\[\text{D = } \begin{array}{c} \includegraphics[width=3cm]{D.png} \end{array}\]

with their derivatives

\[\text{I = } \begin{array}{c} \includegraphics[width=3cm]{I.png} \end{array}\]
\[\text{II = } \begin{array}{c} \includegraphics[width=3cm]{II.png} \end{array}\]
\[\text{III = } \begin{array}{c} \includegraphics[width=3cm]{III.png} \end{array}\]
\[\text{IV = } \begin{array}{c} \includegraphics[width=3cm]{IV.png} \end{array}\]
**THE INTERMEDIATE VALUE THEOREM** Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value $N$ can be taken on once [as in part (a)] or more than once [as in part (b)].

**EXAMPLE 10** Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

**SOLUTION** Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number $c$ between 1 and 2 such that $f(c) = 0$. Therefore, we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Now $f$ is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number $c$ between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root $c$ in the interval $(1, 2)$.

Prove that $x^2 = \sqrt{x+1}$ for some $x$ in the interval $(0, 3)$. State the names of any relevant theorems.