Sphere Rolling Up a Ramp

A solid uniform sphere of mass $M$ and radius $R$ rolls up a ramp of angle $\theta$. There is no slip. At the base of the ramp the velocity of the center of mass is $v_0$ up the ramp. What is the acceleration $a$ of the center of mass and how far up the ramp does the sphere roll before momentarily stopping?

![Diagram of sphere rolling up a ramp with labels $v$, $\theta$, $R$, $n$, $F_s$, $w=Mg$.]

No slip, so:

$\begin{align*}
    v &= R \omega \\
    a &= R \alpha
\end{align*}$

The only torque about $O$ is due to static friction $F_s$ at the point of contact. The sphere must be slowing down so $a$ must be negative for the choice of $x, v,$ and $a$ positive up the ramp. Thus $\alpha$ is also negative for the choice of $\omega$ and $\alpha$ consistent with $v$ and $a$. The net torque $T_o$ must be negative, here counterclockwise. Thus the force of static friction $F_s$ is positive up the ramp.

For a solid sphere,

$I_o = \frac{2}{5}MR^2$
\[ \Sigma F_x = Ma \Rightarrow F_5 - Mgsin\theta = Ma \quad 1. \]
\[ \Sigma T_0 = I_0\alpha \Rightarrow -F_5R = I_0\alpha = \frac{2}{5}MR^2\alpha \]
But \( \alpha = \frac{a}{R} \), so \( F_5 = -\frac{2}{5}Ma \quad 2. \)
Using 1.) and 2.) eliminate \( F_5 \)
\[ \Rightarrow Ma + Mgsin\theta = -\frac{2}{5}Ma \]
\[ \Rightarrow \frac{7}{5}Ma = -Mgsin\theta \]
\[ \Rightarrow a = -\frac{5}{7}gsin\theta \]

This \( a \) is constant, so kinematic equations for constant acceleration may be used, in particular
\[ v^2 = v_0^2 + 2a(x-x_0) \]
At the maximum \( x \) up the ramp with \( x_0 = 0 \) and \( v = 0 \)
\[ 0^2 = v_0^2 + 2\left(-\frac{5}{7}gsin\theta\right)(x-0) \]
gives
\[ x = \frac{7}{10} \frac{v_0^2}{gsin\theta} \]

What if the sphere is initially spinning consistent with \( v = Rw \) but on the ramp there is no friction? Then \( F_0 = 0 \) and the sphere continues to spin at \( \omega_0 = \frac{v_0}{R} \) as its center of mass decelerates.
Then equation 1.) gives \( a = -gsin\theta \) and \( v^2 = v_0^2 + 2a(x-x_0) \)
gives
\[ x = \frac{v_0^2}{2gsin\theta} \]
as the maximum distance up the ramp. Comparing the two cases:

\[
\frac{X_{with\ F}}{X_{without\ F}} = \frac{\frac{7}{10}}{\frac{1}{2}} = \frac{7}{5}
\]

The static friction does no work, but does make it possible for both rotational and translational kinetic energy to be converted to potential energy of gravity. Thus the sphere moves further up the ramp when static friction is present.

Now let's work this problem using Work/Energy.

\[
W_{NC} = \Delta E = E_2 - E_1
\]

State 1 is the sphere at the base of the ramp.
State 2 is the sphere at maximum distance up the ramp, \(x_2\).

\(F_5\) does no work because the instantaneous point of application of \(F_5\) always has zero velocity.

\(n\) does no work because it is perpendicular to motion.

Thus \(W_{NC} = 0\) and thus \(E_2 = E_1\).
Let \( y_1 = 0 \) and then \( y_2 = x_2 \sin \theta \)

\[
\omega_1 = \frac{u_1}{R}, \quad u_1 = u_0
\]

\[
E_1 = K_{rot1} + K_{trans1} + U_{g1}
= \frac{1}{2} I_0 \omega_1^2 + \frac{1}{2} M u_0^2 + M g y_1
= \frac{1}{2} \frac{2}{5} M R^2 \frac{u_0^2}{R^2} + \frac{1}{2} M u_0^2 + M g \cdot 0
= \frac{7}{10} M u_0^2
\]

\[
E_2 = K_{rot2} + K_{trans2} + U_{g2}
= 0 + 0 + M g \cdot x_2 \sin \theta
= M g x_2 \sin \theta
\]

\[
E_2 = E_1 \quad \Rightarrow \quad M g x_2 \sin \theta = \frac{7}{10} M u_0^2
\]

\[
x = X_2 = \frac{7}{10} \frac{u_0^2}{g \sin \theta}
\]

There is nothing in this Work/Energy analysis that requires \( a \) to be constant. If \( a \) is constant (which is true for this problem) then using

\[
u^2 = u_0^2 + 2a (x - x_0)
\]

Gives \( a = -\frac{5}{7} g \sin \theta \)