The Fundamental Theorem of Calculus

THE FUNDAMENTAL THEOREM OF CALCULUS, PART I  If \( f \) is continuous on \([a, b]\), then the function \( g \) defined by
\[
g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b
\]
is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( g'(x) = f(x) \).

This is equivalent to: \( \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x) \)

Simple example:
If \( g(x) = \int_3^x e^{t^2-t} \, dt \)
\[
\frac{dg}{dx} = e^{x^2-x}
\]

Complicated example:
If \( g(x) = \int_{\sqrt{x}}^{x^3} \sin t \, dt \)

Rewrite \( g(x) \) as
\[
g(x) = \int_0^{\sqrt{x}} \sin t \, dt + \int_{\sqrt{x}}^{x^3} \sin t \, dt
\]
\[
= -\int_0^{\sqrt{x}} \sin t \, dt + \int_0^{x^3} \sin t \, dt
\]

Then \( \frac{dg}{dx} = -\sqrt{x} \sin \sqrt{x} \frac{d}{dx} (\sqrt{x}) + \sqrt{x^3} \sin (x^3) \frac{d}{dx} (x^3) \)
\[
= -x^{1/4} \sin \sqrt{x} \sqrt{x} + x^{3/2} \sin (x^3) (3x^2)
\]
\[
\frac{dg}{dx} = \frac{\sin \sqrt{x}}{2 \sqrt{x}} + 3x^{7/2} \sin (x^3)
\]
Note: the independent variable (here \(x\)) must be involved in only the upper limit and if the limit is a function of the independent variable the chain rule is needed.

**THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2** If \(f\) is continuous on \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

where \(F\) is any antiderivative of \(f\), that is, a function such that \(F' = f\).

It is best to add another step:

\[
\int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = F(b) - F(a)
\]

ie first write down an antiderivative function of \(f(x)\) and then evaluate that function at the upper limit and subtract the evaluation at the lower limit.

**Simple example:**

\[
\int_{-1}^{2} (x^3 - 2x) \, dx = \left( \frac{x^4}{4} - x^2 \right) \bigg|_{-1}^{2}
\]

\[
\begin{align*}
&= \frac{2^4}{4} - 2^2 - \left( \frac{(-1)^4}{4} - (-1)^2 \right) \\
&= \frac{16}{4} - 4 - \left( \frac{1}{4} - 1 \right) \\
&= 4 - 4 - \frac{1}{4} + 1 \\
&= \frac{3}{4}
\end{align*}
\]