Decomposition of a Given Vector

Problem: Given vector $\mathbf{v} = (1, -9, 1) \in \mathbb{R}^3$, write $\mathbf{v}$ as $\mathbf{w} + \mathbf{w}^\perp$ where $\mathbf{w}$ is in the subspace $(a, b, a-zb)$ of $\mathbb{R}^3$ and $\mathbf{w}^\perp$ is perpendicular to the subspace $(a, b, a-zb)$.

Defining a subspace in this manner is saying that the subspace consists of all vectors in $\mathbb{R}^3$ which may be written as $a(1,0,1) + b(0,1,-2)$

that is, $\mathbf{d}_1 = (1,0,1)$

and $\mathbf{d}_2 = (0,1,-2)$

are basis vectors for the given subspace. However $\mathbf{d}_1$ and $\mathbf{d}_2$ are neither orthogonal nor unit vectors, since

$\mathbf{d}_1 \cdot \mathbf{d}_2 = -2 \neq 0$

$\| \mathbf{d}_1 \| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 1$

$\| \mathbf{d}_2 \| = \sqrt{1^2 + (-2)^2} = \sqrt{5} \neq 1$

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be orthonormal basis vectors for the same subspace (i.e., both orthogonal and unit vectors).

The first, $\mathbf{u}_1$, may be taken as just $\mathbf{d}_1$ made a unit vector, i.e.

$\mathbf{u}_1 = \frac{\mathbf{d}_1}{\| \mathbf{d}_1 \|} = \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$
To find the $\overline{d}_z$, we first find the vector component of $\overline{d}_z$ in the direction of $\overline{u}_i$, i.e.

\[
(\overline{d}_z \cdot \overline{u}_i) \overline{u}_i
\]

\[
= \left[\left(0, 1, -2\right) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right] \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)
\]

\[
= -\frac{2}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = (-1, 0, -1)
\]

This is subtracted from $\overline{d}_z$, i.e.

\[
\left(0, 1, -2\right) - (-1, 0, -1)
\]

\[
= (1, 1, -1)
\]

and then this is made a unit vector to become $\overline{u}_z$, i.e.

\[
\overline{u}_z = \frac{(1, 1, -1)}{\| (1, 1, -1) \|} = \frac{(1, 1, -1)}{\sqrt{3}}
\]

\[
\overline{u}_z = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)
\]

Clearly $\overline{u}_i$ and $\overline{u}_z$ are unit vectors, and since $\overline{u}_i \cdot \overline{u}_z = 0$, they are orthogonal.

Also $\overline{d}_i = \sqrt{2} \overline{u}_i$.

and $\overline{d}_z = -\sqrt{2} (\overline{u}_i) + \sqrt{3} (\overline{u}_z)$

So $\overline{d}_i$ with $\overline{d}_z$ span the same subspace as do $\overline{u}_i$ with $\overline{u}_z$

i.e.

\[
\text{Span} \left\{ \overline{d}_i, \overline{d}_z \right\} = \text{Span} \left\{ \overline{u}_i, \overline{u}_z \right\}
\]
The desired vector \( \overline{w} \), i.e. the portion of \( \overline{v} \) in the given subspace is then:

\[
\overline{w} = (\overline{v} \cdot \overline{u}_1) \overline{u}_1 + (\overline{v} \cdot \overline{u}_2) \overline{u}_2
\]

\[
= (1,-9,1) \cdot \left( \frac{1}{\nu_2}, 0, \frac{1}{\nu_2} \right) \left( \frac{1}{\nu_2}, 0, \frac{1}{\nu_2} \right) + (1,-9,1) \cdot \left( \frac{1}{\nu_3}, \frac{1}{\nu_3}, -\frac{1}{\nu_3} \right) \left( \frac{1}{\nu_3}, \frac{1}{\nu_3}, -\frac{1}{\nu_3} \right)
\]

\[
= \frac{2}{\nu_2} \left( \frac{1}{\nu_2}, 0, \frac{1}{\nu_2} \right) + \frac{-9}{\nu_3} \left( \frac{1}{\nu_3}, \frac{1}{\nu_3}, -\frac{1}{\nu_3} \right)
\]

\[
= (1, 0, 1) + (-3, -3, 3)
\]

\[
\overline{w} = (-2, -3, 4)
\]

and \( \overline{w}_\perp \) is what remains of \( \overline{v} \) when \( \overline{w} \) is subtracted, i.e.

\[
\overline{w}_\perp = \overline{v} - \overline{w} = (1,-9,1) - (-2,-3,4)
\]

\[
\overline{w}_\perp = (3, -6, -3)
\]

Let's check our result.

Clearly \( \overline{v} = \overline{w} + \overline{w}_\perp \)

and \( \overline{w}_\perp \) is not in the given subspace since

\[
\overline{w}_\perp \cdot \overline{d}_1 = (3, -6, -3) \cdot (1, 0, 1) = 0
\]

and

\[
\overline{w}_\perp \cdot \overline{d}_2 = (3, -6, -3) \cdot (0, 1, -2) = 0
\]