Vector Inner Product

For vectors \( \vec{u} \) and \( \vec{v} \in \mathbb{R}^n \), an inner product (also called a dot product) is denoted as \( \vec{u} \cdot \vec{v} \).

For example, when \( \vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) and \( \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \)

then \[ \vec{u} \cdot \vec{v} = 2(-1) + 3(2) + (-1)(2) = -2 + 6 - 2 = 2 \]

The dot product is a scalar. It may take another form \( \vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} \) and this follows the rules of matrix multiplication.

Some properties are:
\[ \begin{align*}
\vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u} \\
(\vec{u} + \vec{v}) \cdot \vec{w} &= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \\
(c \vec{u}) \cdot \vec{v} &= c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c \vec{v}) \\
\vec{u} \cdot \vec{u} &\geq 0
\end{align*} \]

The norm (length) of vector \( \vec{u} \), denoted \( \| \vec{u} \| \), is defined as \( \| \vec{u} \| = \sqrt{\vec{u} \cdot \vec{u}} \).

This is the familiar hypotenuse length in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) and is the definition of vector length in higher dimensions.

Example: \( \| \vec{v} \| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3 \)

Student: Find \( \| \vec{w} \| \) Answer: \( \sqrt{14} \)

We easily see that \( \| \vec{w} \| = 0 \iff \vec{w} = \vec{0} \).

Unit Vector - If \( \| \vec{u} \| = 1 \), then \( \vec{u} \) is called a unit vector and sometimes denoted as \( \hat{u} \).
Any vector \( \mathbf{u} \) may be turned into an associated unit vector \( \hat{\mathbf{u}} \) by \( \hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} \)

Example: For original \( \mathbf{v} = \begin{bmatrix} -1 \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \)

\( \hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \)

Student: Find \( \hat{\mathbf{u}} \).

The distance between vectors \( \mathbf{u} \) and \( \mathbf{v} \) is

\[
\text{dist}(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \| \mathbf{v} - \mathbf{u} \|
\]

Example: Using our original \( \mathbf{u} \) and \( \mathbf{v} \), we have

\[
\mathbf{u} - \mathbf{v} = \begin{bmatrix} \frac{2}{3} \\ -1 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}
\]

So \( \text{dist}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \| \hat{\mathbf{u}} - \hat{\mathbf{v}} \| = \sqrt{\frac{2}{3}^2 + 1^2 + (-3)^2} = \sqrt{9 + 1 + 9} = \sqrt{19} \)

This is the usual distance used in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) between points located by vectors \( \mathbf{u} \) and \( \mathbf{v} \). This defines distance in higher dimensions.

Orthogonality - Two vectors are said to be orthogonal if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \)

This corresponds to our usual idea of vectors being perpendicular in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)

This illustrates a vector \( \mathbf{u} \) which is orthogonal (perpendicular) to any (and all) vectors in \( W \).

If \( W \) is a two dimensional subspace of \( \mathbb{R}^3 \), then \( W \) is a plane.
The geometric interpretation of the dot product of two vectors is:
\[ \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta) \]
where \( \theta \) is the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \).
Thus
\[ \theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) \]

This is very useful for evaluating angles in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) and is used to define angles in higher dimensional vector space.

Orthogonal Set - If \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthogonal set of nonzero vectors in \( \mathbb{R}^n \), then \( S \) is linear independent and is thus a basis for the subspace \( W \) of \( \mathbb{R}^n \) spanned by \( S \). Being an orthogonal set of vectors means that \( \mathbf{u}_i \cdot \mathbf{u}_j = 0 \) for all \( i \neq j \).

I like to think of orthogonality as "super" independence. Any vector \( \mathbf{y} \) in \( W \) may be uniquely written as a linear combination of the \( \mathbf{u}_i \) vectors in \( S \):
\[ \mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p \]
and \( c_j \) is easily calculated as:
\[ c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \ldots, p) \]

Remember, the \( \mathbf{u}_j \) must belong to an orthogonal set \( S \) to do it this way. One can always solve
\[ A\mathbf{c} = \mathbf{y}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}, \quad A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix} \]
Example: \( \overline{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \overline{u}_2 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \overline{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \overline{y} = \begin{bmatrix} -4 \\ -3 \end{bmatrix} \)

\[
\begin{align*}
\overline{u}_1 \cdot \overline{u}_2 &= -1 + 0 + 1 = 0 \\
\overline{u}_1 \cdot \overline{u}_3 &= 2 + 0 - 2 = 0 \\
\overline{u}_2 \cdot \overline{u}_3 &= -2 + 4 - 2 = 0
\end{align*}
\] Thus \( S = \{ \overline{u}_1, \overline{u}_2, \overline{u}_3 \} \) is an orthogonal set.

\[
\begin{align*}
\overline{y} \cdot \overline{u}_1 &= 8 + 0 - 3 = 5 \quad \Rightarrow \quad c_1 = \frac{5}{2} \\
\overline{u}_1 \cdot \overline{u}_1 &= 1 + 0 + 1 = 2
\end{align*}
\]

\[
\begin{align*}
\overline{y} \cdot \overline{u}_2 &= -8 - 16 - 3 = -27 \quad \Rightarrow \quad c_2 = \frac{-27}{18} = -\frac{3}{2} \\
\overline{u}_2 \cdot \overline{u}_2 &= 1 + 16 + 1 = 18
\end{align*}
\]

\[
\begin{align*}
\overline{y} \cdot \overline{u}_3 &= 16 - 4 + 6 = 18 \quad \Rightarrow \quad c_3 = \frac{18}{9} = 2 \\
\overline{u}_3 \cdot \overline{u}_3 &= 4 + 1 + 4 = 9
\end{align*}
\]

\[
\overline{y} = c_1 \overline{u}_1 + c_2 \overline{u}_2 + c_3 \overline{u}_3 = \frac{5}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}
\]

For Students:
\( \overline{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \overline{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \overline{y} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} \)

- Is \( S = \{ \overline{u}_1, \overline{u}_2 \} \) an orthogonal set?
- Write \( \overline{y} \) as \( \overline{y} = c_1 \overline{u}_1 + c_2 \overline{u}_2 \)

Answers: yes, \( \overline{y} = -\frac{3}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} -2 \\ 6 \end{bmatrix} \)