Planes and Lines Using Some Math 4A

A plane appears as shown in the first octant of $\mathbb{R}^3$. It is parallel to the $y$ axis. What is its algebraic equation?

Three points that are not co-linear define a plane.

Two points are obvious, i.e. $\overline{p}_1 = (0,0,2)$ and $\overline{p}_2 = (3,0,0)$. Since the plane is parallel to the $y$ axis, the $y$ coordinate may be anything as long as the $x$ and $z$ coordinates are correct. So $\overline{p}_3 = (3,1,0)$ is a suitable third point.

Use these three points to determine two independent directions that are parallel to the plane:

$$\overline{u} = \overline{p}_2 - \overline{p}_1 = (3,0,0) - (0,0,2) = (3,0,-2)$$

and

$$\overline{v} = \overline{p}_3 - \overline{p}_1 = (3,1,0) - (0,0,2) = (3,1,-2)$$

Note, we could also use $\overline{p}_3 - \overline{p}_2$ or the negative of each of these directions.

The cross product of two vectors is perpendicular to both vectors. Thus $\overline{n} = \overline{u} \times \overline{v}$ is perpendicular to both $\overline{u}$ and $\overline{v}$ and as such is normal to desired plane.

$$\overline{n} = \overline{u} \times \overline{v} = \begin{vmatrix} i & j & k \\ 3 & 0 & -2 \\ 3 & 1 & -2 \end{vmatrix} = i(0+2) + j(-6+0) + k(3-0) = (2,0,3)$$

Note: $\overline{v} \times \overline{u} = (-2,0,-3)$ and is just as good.

Points $\overline{r} = (x,y,z)$ are in the plane if

$$(\overline{r} - \overline{r}_0) \cdot \overline{n} = 0,$$

where $\overline{r}_0 = (x_0,y_0,z_0)$ is any point known to be in the plane, i.e. $\overline{p}_1$, or $\overline{p}_2$, or $\overline{p}_3$.

A short cut is that for a normal $\overline{n} = (a,b,c)$ then $ax + by + cz = d$ is the equation of the desired plane, with $d$ chosen based on a known point $\overline{r}_0 = (x_0,y_0,z_0)$. 


So our plane is $2x + 0y + 3z = d$, and using $\vec{r}_0 = \vec{p}_1 = (0, 0, 2)$ we find $d = 6$. Check that the other points give the same $d = 6$.

So the plane's algebraic equation is:

$$2x + 0y + 3z = 6 \quad \text{(ie } 2x + 3z = 6, \text{ just remember we are in } \mathbb{R}^3)$$

If given an algebraic equation, a parametric equation of the same plane may be obtained by solving for $x$ or $y$ or $z$ as long as its coefficient is not zero. So here I will solve for $x$. Using Math 4A, we have:

$$\begin{bmatrix} 2 & 0 & 3 & | & 6 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 3/2 & | & 3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}r + \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix}s, \quad r, s \in \mathbb{R}$$

or in current notation,

$$\vec{F} = (x, y, z) = (3, 0, 0) + (0, 1, 0)r + (3/2, 0, 1)s$$

$$\vec{F} = (3 - \frac{3}{2}s, \quad r, \quad s)$$

Note, when picking parameter $r$ we are here choosing $y$ and when picking parameter $s$ we are here choosing $z$ and also $x$ changes since it depends on $z$. A plane is a 2-dimensional subspace of $\mathbb{R}^3$.

Given a parametric equation, to obtain an algebraic equation, just eliminate the parameters, i.e. $r$ and $s$.

So in this case:

$$\begin{align*}
x &= 3 - \frac{3}{2}s \\
y &= r \\
z &= s
\end{align*}$$

$$2x + 3z = 6$$

The $y = r$ does not relate to others so $y$ does not appear in the algebraic equation.
Let's pick a line in the above plane, say one that includes \( \vec{p}_1 \) and \( \vec{p}_3 \) and in parametric form moves from \( \vec{p}_1 \) to \( \vec{p}_3 \) as its parameter \( t \) changes from 0 to 1. The line's direction vector is
\[
\vec{w} = \vec{p}_3 - \vec{p}_1 = (3, 1, 0) - (0, 0, 2) = (3, 1, -2)
\]
The equation is
\[
\vec{l}(t) = (x, y, z) = \vec{p}_1 + t\vec{w} = (0, 0, 2) + (3, 1, -2)t
\]
\[
i.e. \quad \vec{l}(t) = (3t, t, 2-2t), \quad t \in \mathbb{R}^1
\]
Now an algebraic representation of this line. This consists of two simultaneous linear equations. Each is the equation of a plane and the line is their intersection, i.e., points common to both planes. Two planes always intersect if their normal directions are independent. We already have one plane, i.e., \( 2x + 3z = 6 \) with normal \( \vec{n}_1 = (2, 0, 3) \) Remember, the line's direction, i.e., \( \vec{w} = (3, 1, -2) \) is perpendicular to \( \vec{n}_1 \), i.e., \((2, 0, 3) \cdot (3, 1, -2) = 0\).
A second plane must have a normal which is also perpendicular to \( \vec{w} \) and independent of \( \vec{n}_1 \). (It can be but does not need to be perpendicular to \( \vec{n}_1 \).)
The choice is easy, we need a \( \vec{n}_2 = (a_x, b_z, c_z) \) such that \( \vec{n}_2 \cdot (3, 1, -2) = 0 \) and \( \vec{n}_2 \) not a scalar multiple of \( \vec{n}_1 \). Let \( \vec{n}_2 = (1, -3, 0) \) say.
So a second plane is \( x - 3y = d \). It must include a point on the line, for example \( \vec{p}_1 = (0, 0, 2) \).
This means that \( d = 0 \). (Note \( \vec{p}_3 \) also makes \( d = 0 \).
So the second plane is \( x - 3y = 0 \).
So an algebraic representation of the line is:

\[
\begin{align*}
2x + 3z &= 6 \\
x - 3y &= 0
\end{align*}
\]

(Obviously not unique.)

Student should confirm that \( \bar{\lambda} = (3t, t, 2 - 2t) \)

satisfies both of these equations for all \( t \).

From these two equations let's recover a parametric equation for the line. Just solve the set of two equations, as in Math 4A

\[
\begin{bmatrix}
2 & 0 & 3 & \mid & 6 \\
1 & -3 & 0 & \mid & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & \frac{3}{2} & \mid & 3 \\
0 & 1 & \frac{1}{2} & \mid & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
3 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
-\frac{3}{2} \\
-\frac{1}{2} \\
1
\end{bmatrix} t^*
\quad \bar{\lambda}^* = (3 - \frac{3}{2} t^*, 1 - \frac{1}{2} t^*, t^*)
\]

is this same as \( \bar{\lambda} = (3t, t, 2 - 2t) \) ?

Let \( t^* = 2 - 2t \), then \( \bar{\lambda}^* = (3 - \frac{3}{2} (2 - 2t), 1 - \frac{1}{2} (2 - 2t), 2 - 2t) \)

\( \bar{\lambda}^* = (3t, t, 2 - 2t) \)

Same line, the parameters \( t \) and \( t^* \) are scaled and shifted relative to each other.

\( \bar{\lambda}^*(t^*) \) goes from \( \bar{p} \), to \( \bar{p}_3 \) as \( t^* \) goes from 2 to 0.

In previous handout the Algebraic (Vector) equation of a line is \( (\bar{x} - \bar{p}) \times \bar{n} = \bar{o} \), or in current notation \( (\bar{\lambda} - \bar{p}_i) \times \bar{w} = \bar{o} \)

This gives

\[
\begin{align*}
2y + z - 2 &= 0 \\
2x + 3z - 6 &= 0 \\
x - 3y &= 0
\end{align*}
\]

Each of these equations is a linear combination of the other two. So really there are only two independent equations.