Path and Surface Integrals

Green's, Stokes', and Divergence Theorems

Path Integrals

Parameterize path \( \bar{c}(t) = (x(t), y(t), z(t)) \)

* Path integral of a scalar function \( f(x, y, z): \mathbb{R}^3 \rightarrow \mathbb{R} \)
  \[
  \int_{a}^{b} f(\bar{c}(t)) \| \bar{c}'(t) \| \, dt,
  \]
  Set \( f = 1 \) for path length

* Path integral of a vector field \( \bar{F}(x, y, z): \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)
  \[
  \int_{a}^{b} \bar{F}(\bar{c}(t)) \cdot \bar{c}'(t) \, dt
  \]

Surface Integrals

Parameterize surface \( \bar{F}(u, v) = (x(u, v), y(u, v), z(u, v)) \)

Find tangent vectors
\[
\bar{T}_u = \frac{\partial \bar{F}}{\partial u}, \quad \bar{T}_v = \frac{\partial \bar{F}}{\partial v}
\]

Find normal vector
\[
\bar{N}(u, v) = \bar{T}_u \times \bar{T}_v
\]

(This is not a unit normal. It provides both direction and scaling information.)

* Surface integral of a scalar function \( f(x, y, z): \mathbb{R}^3 \rightarrow \mathbb{R} \)
  \[
  \iint_{S} f \, dS = \iint_{D_{u,v}} f(\bar{F}(u, v)) \| \bar{N}(u, v) \| \, du \, dv
  \]
  Set \( f = 1 \) for surface area

* Surface integral of a vector field \( \bar{F}(x, y, z): \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)
  \[
  \iint_{S} \bar{F} \cdot d\bar{S} = \iint_{D_{u,v}} \bar{F}(\bar{F}(u, v)) \cdot \bar{N}(u, v) \, du \, dv
  \]
Green's Theorem applies in $\mathbb{R}^2$

For $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = (P, Q)$, $P$ and $Q \in \mathbb{C}^3$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy$$

Path Integral = Double Integral

$\mathcal{C}$ is a path which forms the boundary of $D$. This is often indicated as $\mathcal{C} = \partial D$

$\mathcal{C}$ goes completely around $D$ but may also act to exclude parts of the interior of $D$. The positive orientation of $\mathcal{C}$ is needed which keeps $D$ toward the left as $\mathcal{C}$ is followed.

Special choices of $\mathbf{F}$ allow Green's Theorem to calculate areas

* When $\mathbf{F} = (-y, 0)$ i.e. $P = -y$ and $Q = 0$
  $$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \text{Area of enclosed } D$$

* When $\mathbf{F} = (0, x)$ i.e. $P = 0$ and $Q = x$
  $$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \text{Area of enclosed } D$$

* When $\mathbf{F} = (-y, x)$ i.e. $P = -y$ and $Q = x$
  $$\frac{1}{2} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \text{Area of enclosed } D$$
Divergence Theorem

\[ \iiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div} \mathbf{F} \, dx \, dy \, dz \]

Surface Integral = Ordinary Triple Integral

S is a surface which completely encloses the region W of \( \mathbb{R}^3 \). The positive orientation of S is away from the region W. Part of S can act to exclude region(s) internal to W from being part of W.

Stokes' Theorem

This applies in \( \mathbb{R}^3 \) and is the same as Green's Theorem. You can think of Stokes' Theorem as the extension of Green's Theorem to \( \mathbb{R}^3 \).

\[ \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \oiint_S (\text{curl} \mathbf{F}) \cdot d\mathbf{S} \]

Path Integral = Surface Integral

Path \( \mathcal{C} \) is the boundary of a surface S. Again, denoted \( \mathcal{C} = \partial S \). The positive direction for \( \mathcal{C} \) is such as to keep the positively oriented surface S on the left of the path.
The surface $S$ in Stokes' Theorem need not enclose a region of $\mathbb{R}^3$. However, if it does completely enclose a region then no path acts as the boundary of that surface $S$. Thus, in this case

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 0 \quad (\text{No } \mathbf{c})$$

and so

$$\iiint_{\mathcal{W}} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = 0$$

This is consistent with the divergence theorem (written with a $\mathcal{G}$) which says:

$$\iiint_{\mathcal{W}} \mathbf{G} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div } \mathbf{G} \, dx\,dy\,dz$$

Let $\mathbf{G} = \text{curl } \mathbf{F}$ and we see that the left side is zero from Stokes.

The right side involves

$$\text{div} \left( \text{curl } \mathbf{F} \right)$$

Which we previously learned is always zero,

$$\text{div}(\text{curl } \mathbf{F}) = \nabla \cdot \nabla \times \mathbf{F} = 0$$