Pivot and Free Variable Columns Redux

Pivot columns and free variable columns are easily identified when a matrix has been placed in either echelon or RREF format. The elementary row operations used to reach echelon or RREF do not move columns of the A matrix. Thus the identity of columns does not change, but the information content of columns does change.

We will characterize each of the columns of the original A matrix as either a pivot or a free variable column identically as they are characterized in the echelon or RREF form.

Each free variable column of the original A matrix is a linear combination of the pivot columns of A located to the left of that free variable column. The multiplier weights used in the linear combination are the numbers in order (ie top down) in the RREF version of the free variable column being considered.
Example:

\[
A = \begin{bmatrix}
1 & -2 & -1 & 2 \\
2 & -4 & -3 & 10 \\
-2 & 4 & 3 & 10 \\
\end{bmatrix}
\]

Because the 1st and 3rd columns of the RREF are pivot columns, the 1st and 3rd columns of \( A \) are identified to be pivot columns. And, since the 2nd and 4th columns of the RREF are free variable columns, the 2nd and 4th columns of \( A \) are identified to be free variable columns.

The second column of \( A \), being a free variable column, is a linear combination of all pivot columns of \( A \) to its left (in this case only one, the 1st column).

\[
\begin{bmatrix}
-2 \\
-4 \\
4 \\
\end{bmatrix} = -2 \begin{bmatrix}
1 \\
2 \\
-2 \\
\end{bmatrix}
\]

The multiplier -2 is obvious, but note it is the number in the same free variable column of the RREF.

Continuing, the 4th column of \( A \), being a free variable column, is a linear combination of all pivot columns of \( A \) to its left. There are two, the
1st and 3rd columns of $A$.

Thus: \[
\begin{bmatrix}
-2 \\
10
\end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\]

and $C_1 = 4$ and $C_2 = 6$, the numbers in the same free variable column of the RREF.

Check: \[
\begin{bmatrix}
-2 \\
10
\end{bmatrix} = 4 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 6 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -18 \\ 18 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}
\]

Problem for students:

Given

\[
A = \begin{bmatrix}
1 & -2 & 0 & 8 \\
-1 & -1 & 3 & -2 \\
1 & 2 & -4 & 0
\end{bmatrix}
\]

Identify the pivot and free variable columns of $A$ and write each free variable column(s) as a linear combination of the pivot column(s).
Answer: RREF is
\[
\begin{bmatrix}
1 & 0 & -2 & 4 \\
0 & 1 & -1 & -2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So
\[
A = \begin{bmatrix}
1 & -2 & 0 & 8 \\
-1 & -1 & 3 & -2 \\
1 & 2 & -4 & 0
\end{bmatrix}
\]

And
\[
\begin{bmatrix}
0 \\
-4
\end{bmatrix} = -2 \begin{bmatrix}
1 \\
1
\end{bmatrix} - 1 \begin{bmatrix}
-2 \\
2
\end{bmatrix}
\]

\[
\begin{bmatrix}
8 \\
-2 \\
0
\end{bmatrix} = 4 \begin{bmatrix}
1 \\
-1
\end{bmatrix} - 2 \begin{bmatrix}
-2 \\
-1 \\
2
\end{bmatrix}
\]