Span, Linear Independence and Basis for Polynomials

The properties of spanning, linear independence and identifying basis sets for polynomials in vector space \( P_n \) can be investigated in terms of the same properties of related vectors in \( \mathbb{R}^{n+1} \) by use of an isomorphism between \( P_n \) and \( \mathbb{R}^{n+1} \). An isomorphism is an invertible mapping which preserves essential information. For example between \( P_2 \) and \( \mathbb{R}^3 \) the usual isomorphism mapping is:

\[
\begin{bmatrix}
a_0 \\ a_1 \\ a_2
\end{bmatrix} \leftrightarrow a_0 + a_1 t + a_2 t^2
\]

Let's investigate the set

\[
S = \{ -t + t^2, 1 + t, 2 + t^2 \}
\]

of polynomials from \( P_2 \).

\[
\begin{align*}
-t + t^2 & \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
1 + t & \leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad 2 + t^2 & \leftrightarrow \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

Thus we will analyse the set \( S' = \{ [0], [1], [2] \} \) of vectors from \( \mathbb{R}^3 \).

We already know how to do this:

Enter the vectors as columns in a matrix \( A \)

\[
A = \begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 1
\end{bmatrix}
\]
Then do EROS to reach an echelon form:

\[
\begin{bmatrix}
0 & 1 & 2 \\
-1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

R1 \leftrightarrow R2  \\
R_2' = R_2 + R_1  \\
R_3' = R_3 - R_2

I have circled the pivots in the echelon form.

- A pivot in every column \(\iff\) column vectors are linear independent
- A pivot in every row \(\iff\) column vectors span \(\mathbb{R}^3\)
- Both linear independent and span \(\iff\) the set of column vectors may be used as a basis set for \(\mathbb{R}^3\).

Because of the isomorphism, the same conclusions are true for the polynomials in \(S\).
- The polynomials are linear independent, they span \(\mathbb{R}_2\), and they constitute a basis set for \(\mathbb{R}_2\).

Let's write the polynomial \(3 - 2t - t^2\) as a linear combination of the polynomials in set \(S\), i.e. we want to find \(x_1, x_2, x_3\) which give

\[3 - 2t - t^2 = x_1(-t + t^2) + x_2(l + t) + x_3(z + t^2)\]

We apply the isomorphism

\[3 - 2t - t^2 \leftrightarrow \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}\]

and the problem becomes

\[
\begin{bmatrix}
0 & 1 & 2 \\
-1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
3 \\
-2 \\
-1
\end{bmatrix}
\]

Augment and RREF, we obtain

\[
\begin{bmatrix}
1 & 0 & 0 & -7 \\
0 & 1 & 0 & -9 \\
0 & 0 & 1 & 6
\end{bmatrix}
\implies
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
-7 \\
-9 \\
6
\end{bmatrix}
\]
Let's check:

\[-7(-t+t^2)-9(1+t) + 6(z+t^2)\]
\[= 7t - 7t^2 - 9t + 12 + 6t^2\]
\[= (-9+12) + (7-9)t + (-7+6)t^2\]
\[= 3 - 2t - t^2 \quad \text{yes}\]

Student: Check spanning, linear independence and determine if a basis set for:

- \(S = \{-t+t^2, 1+t, z+t^2, -1+t\}\)

Answer: Obtain \(A = \begin{bmatrix}
0 & 1 & 2 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}\)

EROS to echelon \(\rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}\)

\(S\) is not linear independent but does span \(P_2\). Not a basis set. Echelon forms are not unique, but you will see a pivot in every row but not in every column for all correct echelon forms.

- \(S = \{1-2t+t^2, -1+t\}\)

Answer: Obtain \(A = \begin{bmatrix}
1 & -1 \\
-2 & 1 \\
1 & 0
\end{bmatrix}\)

EROS to echelon \(\rightarrow \begin{bmatrix}
1 & -1 \\
0 & 1 \\
0 & 0
\end{bmatrix}\)

\(S\) is linear independent, but does not span \(P_2\). Not a basis set.
\[ S = \{ 2 + t + t^2, -t + t^2, 1 + t^2 \} \]

Answer: Obtain \[ A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

EROS to echelon \[ \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \]

\( S \) is not linear independent and does not span \( \mathbb{P}_2 \). Not a basis set.