Dot Product, Cross Product, Lines, Planes

Given two vectors \( \vec{u} = (u_1, u_2, u_3) \), \( \vec{v} = (v_1, v_2, v_3) \)
(other notation: \( \vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \), etc.)

Vector Dot Product: \( \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \), a scalar
Magnitude (Length) of a vector: \( \| \vec{u} \| = \sqrt{\vec{u} \cdot \vec{u}} \)

Vector Cross Product:
\[
\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix} = \hat{i} (u_2 v_3 - u_3 v_2) \\
+ \hat{j} (u_3 v_1 - u_1 v_3) \\
+ \hat{k} (u_1 v_2 - u_2 v_1)
\]
a vector

Do calculation of cross product however you like, I encourage using determinant but evaluate
in place by "basket weave".

Geometrically: \( \vec{u} \cdot \vec{v} = \| \vec{u} \| \| \vec{v} \| \cos \theta \) \( (\theta = \cos^{-1}\left( \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \| \| \vec{v} \|} \right) ) \)
\[
\| \vec{u} \times \vec{v} \| = \| \vec{u} \| \| \vec{v} \| \sin \theta
\]
where \( \theta \) is the (smaller) angle between \( \vec{u} \) and \( \vec{v} \)

Unit Vector: A unit vector has a length of one.
To make any vector a unit vector (obviously with the same direction) just do: \( \frac{\vec{u}}{\| \vec{u} \|} \)

Important Equivalences:
\( \vec{u} \) and \( \vec{v} \) are orthogonal (perpendicular) \( \iff \vec{u} \cdot \vec{v} = 0 \)
\( \vec{u} \) and \( \vec{v} \) are parallel (includes antiparallel) \( \iff \vec{u} \times \vec{v} = 0 \)
\( \vec{u} \times \vec{v} \) is a vector orthogonal to both \( \vec{u} \) and \( \vec{v} \)
Right Hand Rule

Parametric Equation of a Straight Line:
Given two points, \( \vec{p} = (p_1, p_2, p_3) \) and \( \vec{q} = (q_1, q_2, q_3) \)
The line from point \( \vec{p} \) to point \( \vec{q} \) has direction
\( \vec{v} = (v_1, v_2, v_3) = \vec{q} - \vec{p} = (q_1 - p_1, q_2 - p_2, q_3 - p_3) \)
The line has parametric equation
\[ \vec{x}(t) = (x(t), y(t), z(t)) = \vec{p} + t \vec{v}, \quad t \in \mathbb{R} \]
\[ \vec{x}(t) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3) \]

Algebraic Equation of a Plane:
Let direction \( \vec{n} = (a, b, c) \) be normal (perpendicular) to the plane, and point \( \vec{p} = (x_0, y_0, z_0) \) be on the plane.
Want equation for points \( \vec{x} = (x, y, z) \) of the plane.
\( \vec{x} \) must be such that \( \vec{x} - \vec{p} \) is orthogonal to \( \vec{n} \), i.e.
\[ (\vec{x} - \vec{p}) \cdot \vec{n} = 0 \quad \text{(dot product)} \]
\[ (x-x_0)a + (y-y_0)b + (z-z_0)c = 0 \]
\[ ax + by + cz = ax_0 + by_0 + cz_0 = d \]
ie.e \( ax + by + cz = d \) (d evaluated by making point \( \vec{p} \) satisfy equation)
\( \vec{x} - \vec{p} \) lies in plane
requires \( \vec{x} - \vec{p} \) is \( \perp \) to \( \vec{n} \)
so \( (\vec{x} - \vec{p}) \cdot \vec{n} = 0 \)
Parametric Equation of a Plane:
Let \( \vec{u} \) and \( \vec{v} \) be two different vectors which are tangent to the plane. (\( \vec{u} \) and \( \vec{v} \) need not be orthogonal but often are and also not parallel to each other.) Let \( \vec{p} \) be a point in the plane. Then any point \( \vec{x} = (x,y,z) \) in the plane may be written as:

\[
\vec{x} = \vec{p} + \vec{u}t + \vec{v}s, \quad t \text{ and } s \in \mathbb{R}^1
\]

This should be obvious. \( \vec{p} \) gets you to the plane and then values of \( t \) and \( s \) allow you to move around in the plane.

A plane is a two dimensional object which exists in three dimensional space. (Two parameters, \( t \) and \( s \))

How to convert from one description to the other?

Parametric to algebraic:
- Find one point in plane \( \vec{p} = (x_0,y_0,z_0) \)
- Determine normal, \( \vec{n} = (a,b,c) = \vec{u} \times \vec{v} \)
- Use in equation \( ax + by + cz = d \)

\[
d = ax_0 + by_0 + cz_0
\]

Algebraic to parametric:
- Find three points in plane which are not collinear, \( \vec{p}_0, \vec{p}_1, \vec{p}_2 \)
- Let \( \vec{u} = \vec{p}_1 - \vec{p}_0 \) and \( \vec{v} = \vec{p}_2 - \vec{p}_0 \)
- \( \vec{x} = \vec{p}_0 + \vec{u}t + \vec{v}s \)
Algebraic (Vector) Equation of a Line:

For \( \vec{x} = (x,y,z) \) a general point on the line, 
\( \vec{p} = (x_0,y_0,z_0) \) a specific point on the line, 
and \( \vec{n} = (a,b,c) \) a direction for the line, 
we need the line segment \( \vec{x} - \vec{p} \) to always 
be in the direction of \( \vec{n} \) (i.e. parallel) 
\[ \Leftrightarrow (\vec{x} - \vec{p}) \times \vec{n} = \vec{0} \]  
(Three linear equations in \( x,y,z \))

Component of a Vector in a Desired Direction

We want \( l \), the amount of \( \vec{u} \) 
in the direction determined by \( \vec{v} \), 
\( (l \) will be negative if \( \frac{\pi}{2} < \theta \leq \pi \) 
\[ l = ||\vec{u}|| \cos \theta \]
But \( \vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos \theta \)
So \( l = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||} \) (or \( l = \vec{u} \cdot \vec{v} \) if \( \vec{v} \) a unit vector)

Thus given three non-zero orthogonal vectors \( \vec{u}, \vec{v}, \vec{w} \) 
(\( \vec{u} \cdot \vec{v} = 0, \vec{u} \cdot \vec{w} = 0, \vec{v} \cdot \vec{w} = 0 \)) any vector \( \vec{a} \) may be 
written in terms of components in the directions 
of \( \vec{u}, \vec{v}, \) and \( \vec{w} \) as:
\[ \vec{a} = \frac{\vec{a} \cdot \vec{u}}{||\vec{u}||} \frac{\vec{u}}{||\vec{u}||} + \frac{\vec{a} \cdot \vec{v}}{||\vec{v}||} \frac{\vec{v}}{||\vec{v}||} + \frac{\vec{a} \cdot \vec{w}}{||\vec{w}||} \frac{\vec{w}}{||\vec{w}||} \]

Areas: Two vectors define a parallelogram 
as well as a triangle, and areas are easily 
computed:
\[ A = ||\vec{u} \times \vec{v}|| \]
\[ A = \frac{1}{2} ||\vec{u} \times \vec{v}|| \]