Vector Spaces and Subspaces

Vector Space:

A vector space \( V \) is a nonempty collection of objects called vectors for which are defined the operations

- **vector addition**, denoted \( \vec{x} + \vec{y} \), and
- **scalar multiplication** (multiplication by a real constant), denoted \( c\vec{x} \),

that satisfy the following properties for all \( \vec{x}, \vec{y}, \vec{z} \in V \) and \( c, d \in \mathbb{R} \).

**Closure Properties:**
1. \( \vec{x} + \vec{y} \in V \).
2. \( c\vec{x} \in V \).

**Addition Properties:**
3. There is a zero vector \( \vec{0} \) in \( V \) such that \( \vec{x} + \vec{0} = \vec{x} \). (Additive Identity)
4. For every vector \( \vec{x} \in V \), there is a vector \( -\vec{x} \) in \( V \) (its negative) such that \( \vec{x} + (-\vec{x}) = \vec{0} \). (Additive Inverse)
5. \( (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}) \). (Associativity)
6. \( \vec{x} + \vec{y} = \vec{y} + \vec{x} \). (Commutativity)

**Scalar Multiplication Properties:**
7. \( 1\vec{x} = \vec{x} \). (Scalar Multiplicative Identity)
8. \( c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y} \). (First Distributive Property)
9. \( (c + d)\vec{x} = c\vec{x} + d\vec{x} \). (Second Distributive Property)
10. \( c(d\vec{x}) = (cd)\vec{x} \). (Associativity)

Vector Subspaces:

Vector Subspace Theorem

A nonempty subset \( W \) of a vector space \( V \) is a subspace of \( V \) if it is closed under addition and scalar multiplication:

(i) If \( \vec{u}, \vec{v} \in W \), then \( \vec{u} + \vec{v} \in W \).
(ii) If \( \vec{u} \in W \) and \( c \in \mathbb{R} \), then \( c\vec{u} \in W \).

The Zero-Space Check:

The zero-space \( \{ \vec{0} \} \) is always a subspace of any vector space. If \( \vec{0} \) is not in \( W \), then \( W \) is empty and is not a subspace.
Some problems:

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ be a vector in $\mathbb{R}^4$. The vector $z = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$ is a linear combination of $x_1$ and $x_2$. The set $S = \{ x \in \mathbb{R}^4 : x_1 + x_2 = 0 \}$ is a subspace of $\mathbb{R}^4$.

Take the form $[x_1, x_2, x_3, x_4] = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

For $n \in \mathbb{N}$, is $\mathbb{R}^n$ a vector space?

Note also, $x = 0$ is the zero vector.

If $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then $cx$ is also a vector in $\mathbb{R}^n$.

By definition, $x$ is a vector in $\mathbb{R}^n$.

The vector $0$ is the zero vector of $\mathbb{R}^n$.

The scalar $c$ is used to scalar multiply any vector $x$.

For immediate results are:

\[\text{Span} \{ X, Y \} \text{ is a vector space of } \mathbb{R}^2.\]

The vector $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are linearly independent.

The vector $z = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a linear combination of $X$ and $Y$.

The set $S = \{ x \in \mathbb{R}^2 : x_1 + x_2 = 0 \}$ is a subspace of $\mathbb{R}^2$.

All planes in $\mathbb{R}^3$ passing through the origin $O$.

The only possible subspaces of $\mathbb{R}^2$ are:

1. $\{0\}$
2. $\mathbb{R}^2$