Vector Spaces and Subspaces

Vector Space:

A vector space \( V \) is a nonempty collection of objects called vectors for which are defined the operations

* vector addition, denoted \( \vec{x} + \vec{y} \), and
* scalar multiplication (multiplication by a real constant), denoted \( c \vec{x} \),

that satisfy the following properties for all \( \vec{x}, \vec{y}, \vec{z} \in V \) and \( c, d \in \mathbb{R} \).

Closure Properties:
1. \( \vec{x} + \vec{y} \in V \).
2. \( c \vec{x} \in V \).

Addition Properties:
3. There is a zero vector \( \vec{0} \) in \( V \) such that \( \vec{x} + \vec{0} = \vec{x} \). (Additive Identity)
4. For every vector \( \vec{x} \in V \), there is a vector \( -\vec{x} \) in \( V \) (its negative) such that \( \vec{x} + (-\vec{x}) = \vec{0} \). (Additive Inverse)
5. \( (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}) \). (Associativity)
6. \( \vec{x} + \vec{y} = \vec{y} + \vec{x} \). (Commutativity)

Scalar Multiplication Properties:
7. \( 1 \vec{x} = \vec{x} \). (Scalar Multiplicative Identity)
8. \( c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y} \). (First Distributive Property)
9. \( (c + d)\vec{x} = c\vec{x} + d\vec{x} \). (Second Distributive Property)
10. \( c(d\vec{x}) = (cd)\vec{x} \). (Associativity)

Vector Subspaces:

Vector Subspace Theorem
A nonempty subset \( W \) of a vector space \( V \) is a subspace of \( V \) if it is closed under addition and scalar multiplication:

(i) If \( \vec{u}, \vec{v} \in W \), then \( \vec{u} + \vec{v} \in W \).
(ii) If \( \vec{u} \in W \) and \( c \in \mathbb{R} \), then \( c\vec{u} \in W \).

The Zero-Space Check:
The zero-space \( \{\vec{0}\} \) is always a subspace of any vector space. If \( \vec{0} \) is not in \( W \), then \( W \) is empty and is not a subspace.
To be a subspace of \( \mathbb{R}^3 \) (Yes, \( \mathbb{R}^3 \) is formally considered a subspace of \( \mathbb{R}^3 \)),

- All planes in \( \mathbb{R}^3 \) containing the origin are considered.
- All lines in \( \mathbb{R}^3 \) passing through the origin are.

The single vector \( \mathbf{0} \) is the set \{\mathbf{0}\}.

The only proper subspace of \( \mathbb{R}^3 \) are:

(1) \( \mathbb{R} \) as a subspace of \( \mathbb{R}^3 \) by the operation of linear/coordinate extraction.

(2) \( \mathbb{R} \) as a subspace of \( \mathbb{R}^3 \) by the inclusion of a vector.

(3) All planes in \( \mathbb{R}^3 \) containing the origin.

(4) All lines in \( \mathbb{R}^3 \) passing through the origin.

(5) The single vector \( \mathbf{0} \).

(6) The set \{\mathbf{0}\}.

Some problems:

\[
\begin{bmatrix}
2 \\
3 \\
-1
\end{bmatrix}
\]

is a vector in \( \mathbb{R}^3 \).

\[
\begin{bmatrix}
2 \\
3 \\
-1
\end{bmatrix}
= \begin{bmatrix}
2 \\
3 \\
-1
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

...