Determinant Using EROS

First some properties of determinants:

- $|A^T| = |A|
- |AB| = |A||B| = |B||A|
- $|A^{-1}| = \frac{1}{|A|}$, if $|A| \neq 0$
- $A^{-1}$ exists $\iff |A| \neq 0$
- If $A$ is a triangular matrix (either upper or lower triangular) then $|A| = \text{Product of Entries on the main diagonal.}$

We have learned how to use EROS to change a matrix to its row equivalent echelon form. The echelon form is a (upper) triangular matrix.

The question then is how does the application of EROS change the determinant of a matrix? Let's start with start with a matrix $A$, apply a single EROS, and obtain matrix $B$. Then the effects on the determinant are:

- Replacement, $R_i^* = R_i + cR_j$, $|A| = |B|$
- Interchange, $R_i \leftrightarrow R_j$, $|A| = -|B|$
- Scaling, $R_i^* = \frac{1}{k}R_i$, $|A| = k|B|$

Using EROS, any matrix $A$ can be changed to echelon form by using only row replacement and row interchange. However, row scaling often makes the arithmetic easier.
The impact of row scaling can be remembered by realizing that when all entries of a single row are made smaller by a factor $k$, then the determinant of the altered matrix is also smaller by the same factor. So the desired determinant of the original matrix must be the factor $k$ larger than the determinant of the altered matrix.

Example:

$$A = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & -1 \end{bmatrix}$$

$R_2^* = R_2 - 3R_1$

$R_3^* = R_3 - 2R_1$

det no change

$R_2^* = \frac{R_2}{6}$

det smaller by factor 6

$$\rightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} = B \quad |B| = 1(-3)(1) = -3$$

$R_3^* = R_3 + R_2$

det no change

Thus $|A| = 6|B| = 6(-3) = -18$

Student:

$$\text{Given } A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 2 & -4 & -1 \\ -2 & -6 & 2 & 3 \\ 3 & 7 & -3 & 8 \end{bmatrix}$$

Evaluate $|A|$ using EROS to make a row equivalent upper triangular form. You can do this using three row replacements and one row swap. Answer $|A| = 24$
The following example is a small variation on the previous:

Let \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \) and given that \( |A| = 7 \)

What is \( |B| \), where \( B = \begin{bmatrix} 2d-g & 2e-h & 2f-i \\ a & b & c \\ g & h & i \end{bmatrix} \)

To change \( A \) to \( B \) using EROs, we see that one row interchange, one row scaling increasing by a factor of 2, and one row replacement is needed.

Thus \( |B| = (-)(2)|A| = -14 \)

Notice here we have \( |A| \) and want \( |B| \), the determinant of the altered matrix \( B \). 
\( B \) is scaled up as compared to \( A \).

Determinants can be used to find the area of parallelograms in \( \mathbb{R}^2 \):

\[
\text{Area} = \left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right| = |\det [\vec{u} \, \vec{v}]|.
\]
If the parallelogram is not located with a vertex at the origin, you need to translate it so that the generating sides $\vec{u}$ and $\vec{v}$ do originate at the origin.

Student Problem: Find the area of a parallelogram with vertices:
$$(-1,-2), (3,1), (4,4), \text{ and } (0,1).$$

Answer: 9

Even more amazing is that the volume of a parallelepiped in $\mathbb{R}^3$ works the same way:

$$\text{Volume}(\vec{u}, \vec{v}, \vec{w}) = |\det [\vec{u} \ \vec{v} \ \vec{w}]|.$$