Determinant of a Matrix

The determinant of a matrix is commonly denoted in either of two ways:

\[ \text{Determinant of } A = \det A = |A| \]

Note that the vertical lines are not brackets.

One method for evaluating a determinant is called expansion by cofactors.
Let the \( n \times n \) matrix \( A \) consist of entries \( a_{ij} \). The value of \( i \) is the row number and the value of \( j \) is the column number for the entry location.

Expansion by the 1st row is:

\[ |A| = \det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n} \]

\( C_{ij} \) is the \((i,j)\) cofactor of \( A \) given by

\[ C_{ij} = (-1)^{i+j} \det A_{ij} = (-1)^{i+j} |A_{ij}| \]

where \( A_{ij} \) is an \((n-1) \times (n-1)\) matrix obtained from \( A \) by deleting the \( i \)th row and the \( j \)th column from \( A \).

The value of the determinant is unique for any square matrix \( A \). The same process can be applied expanding by any one row or any one column.

For example, expanding by the 3rd column of \( A \) (assuming \( n \geq 3 \)) gives:

\[ |A| = a_{13} C_{13} + a_{23} C_{23} + \cdots + a_{n3} C_{n3} \]
This expansion uses the already specified C_{ij}.

We will see that this is both very simple in concept, but very intensive in computation as \( n \) becomes larger.

As an example, find the determinant of:

\[
A = \begin{bmatrix}
3 & 1 & -1 \\
2 & 1 & 3 \\
0 & 1 & 2
\end{bmatrix}
\]

Expanding by the 1st row we have:

\[
|A| = (3)(-1)^{1+1} \left| \begin{array}{cc}
1 & 3 \\
0 & 2
\end{array} \right| + (1)(-1)^{1+2} \left| \begin{array}{cc}
2 & 3 \\
0 & 2
\end{array} \right| + (-1)(-1)^{1+3} \left| \begin{array}{cc}
2 & 3 \\
0 & 1
\end{array} \right|
\]

\[
= 3(-1)^2(2-3) + 1(-1)^3(4-0) - 1(-1)^4(2-0)
\]

\[
= 3(1)(-1) + 1(-1)(4) - 1(1)(2)
\]

\[
= -3 - 4 - 2 = -9
\]

\[|A| = -9\]

Let's expand by the 3rd column:

\[
|A| = (-1)(-1)^{1+3} \left| \begin{array}{cc}
2 & 3 \\
0 & 1
\end{array} \right| + 3(-1)^{2+3} \left| \begin{array}{cc}
3 & 1 \\
0 & 1
\end{array} \right| + 2(-1)^{3+3} \left| \begin{array}{cc}
3 & 1 \\
0 & 1
\end{array} \right|
\]

\[
= -1(-1)^4(2-0) + 3(-1)^5(3-0) + 2(-1)^6(3-2)
\]

\[
= -1(1)(2) + 3(-1)(3) + 2(1)(1)
\]

\[
= -2 - 9 + 2 = -9
\]

Same answer

Take a closer look at the effect of \((-1)^{i+j}\).

It is either a +1 or a -1. It is +1 at the \( a_{ii} \) location and alternates as you move down or over into the matrix array to your specific \( a_{ij} \) (never moving diagonally).
We can think of this as either "keep the sign" or "change the sign" applied to a specific $a_{ij}$.

Also, once we reach a determinant of a 2x2 matrix we do it as $|ab| = ad - bc$.

One more time, let’s expand by the second row, taking some obvious short cuts with what we write down:

$$|A| = -2 \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= -2(3) + 1(6) - 3(3)$$

$$= -6 + 6 - 9 = -9.$$ 

Get it? Not so bad!

Student evaluate: $\begin{vmatrix} 1 & 2 & -4 \\ -1 & 2 & 0 \\ 2 & 1 & 5 \end{vmatrix}$ by two different expansions.

Answer $|A| = 40$

This is a recursive process, in that the determinant of an $n \times n$ matrix is reduced to the need to evaluate determinants of $(n-1) \times (n-1)$ matrices and etc.

Notice that to evaluate the determinant of, for example, a 5x5 requires determinants of 5 separate 4x4 matrices, each of which requires 4 separate 3x3 matrices, each of which requires 3 separate 2x2 !!

Thus a total of 60 determinants of 2x2 matrices are involved!! Help!
First some help with determinants of $3 \times 3$ matrices. These will be used a lot in Math 6A.

The procedure is called "Basket Weave." First, the first and second columns of $A$ are written again at the right side of your matrix, giving a $3 \times 5$ array. Then three term products are taken along the diagonals. Products taken along the downward arrows in the coming example are added and products taken along the upward arrows are subtracted. This process only applies to $3 \times 3$ matrices.

Using the previous student problem, we have:

\[
\begin{vmatrix}
-16 & 0 & -10 \\
1 & 2 & 1 \\
-1 & 2 & 0 \\
2 & 1 & 5 \\
0 & 1 & 0 \\
4 & 2 & 1 \\
\end{vmatrix}
\]

\[
\det A = (10 + 0 + 4) - (-16 + 0 - 10) = 14 - (-26) = 40
\]

**Student:** Given $A = \begin{bmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 4 \end{bmatrix}$ what is $|A|$?

**Answer:** 3

With a little practice and determination (to take determinants) you can do this pattern of products without rewriting the two columns.