Damped Oscillator

The damper is like a shock absorber on a car. It produces a force proportional to the velocity of the mass m.

\[ F_{damping} = -b \dot{v}_x \]

Free body diagram in x direction

\[ F_{damping} \quad m \ddot{x} = \sum F_x \]

\[ m \ddot{x} = -kx - b \dot{x} \]

Using \( v_x = \dot{x}, \quad a_x = \ddot{x} \), we have

\[ m \ddot{x} + b \dot{x} + kx = 0 \]

What is called critical damping happens when

\[ b = b_{crit} = 2 \sqrt{mk} \]

An oscillator with \( b < b_{crit} \) is said to be under damped. The response is an oscillation which decreases with time.

\[ x(t) = A_+ e^{-\frac{b}{2m} t} \cos(\omega' t + \phi) \] (transient)

\[ x(t) = A_+ e^{-\frac{b}{2m} t} \cos(\omega' t + \phi) \] (response)

\( A_+ \) and \( \phi \) depend on the initial conditions, i.e. \( x \) and \( \dot{x} \) at \( t = 0 \).

\[ \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \]
When $b$ is "light damping",

\[ \frac{b^2}{4m^2} \] is small as compared to $\frac{k}{m}$

Then $\omega' = \omega_0 = \sqrt{\frac{k}{m}}$ the system's natural frequency.

Notice that if $b = b_{crit}$ we have $\omega' = 0$.
Thus for $b$ larger than critical damping, there is absolutely no oscillation, and the system is said to be overdamped. The response in general is then the sum of two decaying exponentials.

1. Underdamped case: The system oscillates around the equilibrium position, and the amplitude continually decreases.

2. Critically damped case: The system returns to equilibrium without overshooting. There are no oscillations around equilibrium.

3. Overdamped case: The system returns to equilibrium without overshooting, but more slowly than if critically damped. There are no oscillations around equilibrium.

Figure 12.19 Underdamped, critically damped, and overdamped oscillations The motion of a damped oscillating system depends on whether it is underdamped ($b < 2 \sqrt{km}$), critically damped ($b = 2 \sqrt{km}$), or overdamped ($b > 2 \sqrt{km}$). In each case the system is displaced from equilibrium by a distance $A$ and then released.
The transient response $x_t$ discussed above always decays with time and eventually is undetectable.

A forced damped oscillator adds some external force, $F(t)$.

The total response is the sum of the transient response and the forced response caused by the forcing function $F(t)$. Since the transient response decays away, the forced response is the only remaining after sufficient time.

For the case of $F(t) = F_0 \cos(\omega t)$, $F_0$ is the amplitude and $\omega$ is its frequency. (Remember, we used $\omega_0$ for $\sqrt{k/m}$, so $\omega$ is any desired angular frequency.)

The system's forced response, (also called steady state response) is

$$x_{\text{steady state}} = A \cos(\omega t + \phi)$$

That is, it follows the forcing function at exactly the same $\omega$ as the forcing function but with a phase shift (different $\phi$ than that used in transient response).
The amplitude $A$ of this steady state response is

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}$$

Notice that the denominator becomes smaller when $\omega$ approaches $\omega_0$, i.e., when the frequency of forcing function approaches the natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$ of the spring-mass system. This corresponds to the peaks of the following diagram.

Greater damping (larger $b$):
- Peak becomes broader
- Peak becomes less sharp
- Peak shifts toward lower frequencies

If $b \geq \sqrt{2km}$, peak disappears completely