Optimization With Constraint
Wish to find the maximum or minimum
value of \( f(\mathbf{x}) \) subject to the
constraint that \( g(\mathbf{x}) = k \).

Method: Form the vector equation
\[
\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})
\]
where \( \lambda \) is a new variable called a
Lagrange multiplier.
For \( \mathbf{x} \in \mathbb{R}^n \), this results in \( n \) equations.
Solve these \( n \) equations along with
\[
g(\mathbf{x}) = k
\]
for the \( n+1 \) unknowns, i.e., for \( x_1, \ldots, x_n, \lambda \).
In general there will be a number of
candidate solutions \( \mathbf{x}_i, \lambda_i \).
Evaluate the \( f(\mathbf{x}) \) at all candidate \( \mathbf{x}_i \)
and choose the largest and smallest.
Example Sect 4.4 #11

Find extrema of \( f(x, y) = 3xy \) subject to the constraint \( x^2 + y^2 = 4 \).

So \( f(x) = 3xy \), \( g(x) = x^2 + y^2 \).

\[
\nabla f(x) = \lambda \nabla g(x) \implies (3y, 3x) = \lambda (2x, 2y)
\]

Thus must solve:
1. \( 3y = 2\lambda x \)
2. \( 3x = 2\lambda y \)
3. \( x^2 + y^2 = 4 \)

Eq. 1. gives \( \lambda = \frac{3y}{2x} \).

Enter this in Eq. 2.

\[ 3x = 2\left(\frac{3y}{2x}\right)y \]

and obtain \( 3x = \frac{3y^2}{x} \implies x^2 = y^2 \).

This with Eq. 3 gives \( y^2 + y^2 = 4 \implies y^2 = 2 \).

So \( y = \pm\sqrt{2} \) or \( y = -\sqrt{2} \).

These in Eq. 4 give

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \lambda )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\sqrt{2}, \sqrt{2}))</td>
<td>(3/2)</td>
<td>6</td>
</tr>
<tr>
<td>((\sqrt{2}, -\sqrt{2}))</td>
<td>(-3/2)</td>
<td>-6</td>
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</tbody>
</table>

Maximum = 6
Minimum = -6
Example Sect 4.4 #17

Find extrema of \( f(x, y, z) = x + 2y - 4z \)
subject to the constraint \( x^2 + y^2 + 2z^2 = 4 \).
Note: the level surfaces of \( f \) are planes and the constraint is an ellipsoid.
So \( f(x) = x + 2y - 4z \), \( g(x) = x^2 + y^2 + 2z^2 \)

\[ \nabla f(x) = \lambda \nabla g(x) \rightarrow (1, 2, -4) = \lambda (2x, 2y, 4z) \]
Thus the 4 equations to solve simultaneously are:

\[ \begin{align*}
1 &= \lambda 2x \\
2 &= \lambda 2y \\
-4 &= \lambda 4z \\
x^2 + y^2 + 2z^2 &= 4
\end{align*} \]

Obtain \( x = \frac{1}{2\lambda} , y = \frac{1}{\lambda} , z = -\frac{1}{\lambda} \)
put into the 4th equation
\[ \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{2}{\lambda^2} = 4 \rightarrow \frac{1}{\lambda^2} (\frac{13}{4} + 1 + 2) = 4 \]
\[ \frac{1}{\lambda^2} (\frac{13}{4}) = 4 \rightarrow \lambda^2 = \frac{13}{16} , \lambda = +\frac{\sqrt{13}}{4} , \lambda = -\frac{\sqrt{13}}{4} \]
So \( \lambda = \frac{\sqrt{13}}{4} , x = \frac{2}{\sqrt{13}} , y = \frac{4}{\sqrt{13}} , z = -\frac{4}{\sqrt{13}} \)

\[ f(x, y, z) = \frac{2}{\sqrt{13}} + \frac{8}{\sqrt{13}} + \frac{16}{\sqrt{13}} = \frac{26}{\sqrt{13}} \text{ (Book error)} \]

And when \( \lambda = -\frac{\sqrt{13}}{4} , x = -\frac{2}{\sqrt{13}} , y = -\frac{4}{\sqrt{13}} , z = \frac{4}{\sqrt{13}} \)

\[ f(x, y, z) = -\frac{2}{\sqrt{13}} - \frac{8}{\sqrt{13}} - \frac{16}{\sqrt{13}} = -\frac{26}{\sqrt{13}} \]

Maximum = \( \frac{26}{\sqrt{13}} \), Minimum = \( -\frac{26}{\sqrt{13}} \)
Example Sect 4.4 § 25
Find the minimum of the function
\[ f(x,y,z) = x^2 + y^2 + z^2 \] subject to the constraints
\[ 2y + z = 6 \]
and \[ x - 2y = 4 \]

Multiple constraints are handled with additional Lagrange multipliers (see page 269 of textbook)
We have \[ f(x,y,z) = x^2 + y^2 + z^2 \]
\[ g_1(x,y,z) = 2y + z \]
\[ g_2(x,y,z) = x - 2y \]

\[ \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \]
\[ ie \ (2x, 2y, 2z) = \lambda_1 (0, 2, 1) + \lambda_2 (1, -2, 0) \]
These three equations plus the two constraint equations are a system of 5 linear equations.

\[
\begin{bmatrix}
2 & 0 & 0 & -1 & \lambda_1 \\
0 & 2 & 0 & -2 & \lambda_2 \\
0 & 0 & 2 & -1 & 0 \\
0 & 2 & 1 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\lambda_1 \\
\lambda_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
4 \\
\end{bmatrix}
\]

This system may easily be solved by the methods learned in Math 4A
obtaining \[ x = \frac{14}{9}, \ y = \frac{17}{9}, \ z = \frac{16}{9}, \ \lambda_1 = \frac{92}{9}, \ \lambda_2 = \frac{88}{9} \]

\[ f(x,y,z) \] evaluates to \[ f(x,y,z) = 50.22 \]
What we have found is the minimum distance squared between the origin and the line of intersection of two planes.