Using Gradient to Find Normal

Using $\nabla$ to find perpendicular to curves in $\mathbb{R}^2$

If given as $y = f(x)$ work with $F(x,y) = y - f(x) = 0$
If given as $F(x,y) =$ Constant work with $F(x,y)$

Direction vector (in $\mathbb{R}^2$) of a line perpendicular at point $\vec{p} = (x_0, y_0)$ is $\vec{n} = \nabla F(\vec{p})$

The perpendicular line has parametric form

$$\vec{c}(t) = \vec{p} + t \vec{n}$$

The tangent line satisfies

$$(\vec{x} - \vec{p}) \cdot \vec{n} = 0 \quad (\vec{x} = (x, y))$$

Which for $\vec{n} = (a, b)$ is

$$ax + by = d$$

Choose $d$ to satisfy point $\vec{p}$.

Example

$$y = x^2, \quad \vec{p} = (2, 4)$$

$$F(x,y) = y - x^2 \quad \nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) = (-2x, 1)$$

$$\nabla F(\vec{p}) = \nabla F(2, 4) = (-4, 1) \equiv \vec{n}$$

Perpendicular line

$$\vec{c}(t) = \vec{p} + t \vec{n} = (2, 4) + t (-4, 1)$$

$$(x, y) = (2, 4) + t (-4, 1)$$

i.e. $x = 2 - 4t, \quad y = 4 + t$

If desired, eliminate $t$, giving $y = -\frac{x}{4} + \frac{9}{2}$

Tangent line for $\vec{n} = (a, b) = (-4, 1)$

is $-4x + 1y = d$

fit to $\vec{p} = (2, 4)$ i.e. $-4 \cdot 2 + 1 \cdot 4 = -y = d$

$$-4x + y = 4 \quad \Rightarrow \quad y = 4x - 4$$
Notice that the slopes are negative reciprocals and both lines pass through \( \bar{p} \).

But what if \( y = x^2 \) were written as \( F(x, y) = \frac{y}{x^2} = 1 \) \((x \neq 0)\)?

We should be able to work with

\[ F(x, y) = \frac{y}{x^2} \]

and obtain the same results!

\[
\vec{DF} = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) = \left( -\frac{2y}{x^3}, \frac{1}{x^2} \right)
\]

It looks very different!

\[
\vec{n} = \vec{DF}(\bar{p}) = \left( -\frac{2 \cdot 4}{2^3}, \frac{1}{2^2} \right) = ( -1, \frac{1}{4} )
\]

But this \( \vec{n} \) is just the previous normal divided by 4, so the exact same perpendicular and tangent lines will result. (Please confirm)

\[ \checkmark \]

Using \( \vec{V} \) to find normal line and tangent plane to a surface in \( \mathbb{R}^3 \)

If surface is given as \( z = f(x, y) \), work with \( F(x, y, z) = z - f(x, y) = 0 \). If surface is given as \( F(x, y, z) = \text{Constant} \), work with \( F(x, y, z) \).

Direction vector in \( \mathbb{R}^3 \) which is normal to surface at point \( \bar{p} (x_0, y_0, z_0) \) (on surface) is \( \vec{n} = \vec{DF}(\bar{p}) \)
The perpendicular line has parametric form
\[ \overrightarrow{a}(t) = \overrightarrow{p} + t \overrightarrow{n} \]
The tangent plane satisfies
\[ (\overrightarrow{x} - \overrightarrow{p}) \cdot \overrightarrow{n} = 0 \quad (\overrightarrow{x} = (x, y, z)) \]
which for \( \overrightarrow{n} = (a, b, c) \) is
\[ ax + by + cz = d \]
Choose \( d \) to satisfy point \( \overrightarrow{p} \).

Example
For surface defined by: \( z = x^2 + y^2 \) find the
direction \( \overrightarrow{n} \) which is normal (perpendicular) to the
surface at point \( \overrightarrow{p} = (x_0, y_0, z_0) = (1, 1, 2) \). Then
find the parametric equation of a line thru \( \overrightarrow{p} \) with
direction \( \overrightarrow{n} \) and also find the algebraic equation
of a plane which is tangent to the surface at \( \overrightarrow{p} \).
The surface cuts the \( x-z \) plane (i.e. \( y=0 \))
with line \( z = x^2 \), thus having the shape of a parabola.
The surface also cuts the \( y-z \) plane (i.e. \( x=0 \))
with a parabola shape, now \( z = y^2 \).
The surface is a paraboloid of revolution about
the \( z \) axis.

To find \( \overrightarrow{n} \), form
\[ F(x, y, z) = z - f(x, y) \]
\[ F(x, y, z) = z - x^2 - y^2 = 0 \]
\[ z = f(x, y) = x^2 + y^2 \]
\[ \overrightarrow{n} = \nabla F(\overrightarrow{p}) \]
\[ \overrightarrow{n} = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \]
\[ \overrightarrow{n} = (-2x, -2y, 1) \]
at point \( \overrightarrow{p} \), \( \overrightarrow{n} = (-2 \cdot 1, -2 \cdot 1, 1) = (-2, -2, 1) \)
Notice that when \( F(x, y, z) = z - f(x, y) \), the resulting normal direction \( \vec{n} \) has a \( z \) component which is always of value 1.

The equation of the desired line normal to the surface at \( \vec{p} \) is:

\[
\vec{z}(t) = (x(t), y(t), z(t)) = \vec{p} + t \vec{n} = (1, 1, 2) + t(-2, -2, 1)
\]

Thus \( x(t) = 1 - 2t \), \( y(t) = 1 - 2t \), \( z(t) = 2 + t \)

The tangent plane at \( \vec{p} \) is:

\[
(\vec{x} - \vec{p}) \cdot \vec{n} = 0
\]

\[
\begin{align*}
&= (x - 1, y - 1, z - 2) \cdot (-2, -2, 1) = 0 \\
&= -2(x - 1) - 2(y - 1) + 1(z - 2) = 0 \\
&= -2x + 2y + 2z + 2 - 2 = 0
\end{align*}
\]

\[
\vec{n} = (-2, -2, 1) = (a, b, c)
\]

Since \( \vec{n} = (-2, -2, 1) \), the tangent plane is immediately known to be \(-2x - 2y + 1z = d\)

and to satisfy \( \vec{p} = (1, 1, 2) \)

\[
-2 \cdot 1 - 2 \cdot 1 + 1 \cdot 2 = d = -2
\]

So \(-2x - 2y + z = -2\)

or \( z = 2x + 2y - 2 \)

The surface could have been described as:

\[
F(x, y, z) = \frac{x^2 + y^2}{2} = 1, \quad z \neq 0
\]

Giving \( \nabla F = \left( \frac{2x}{2}, \frac{2y}{2}, -\frac{(x^2 + y^2)}{2} \right) \)

at \( \vec{p} = (1, 1, 2) \)

\[
\nabla F = \left( \frac{2 \cdot 1}{2}, \frac{2 \cdot 1}{2}, -\frac{(1^2 + 1^2)}{2} \right) = (1, 1, -\frac{1}{2})
\]

This normal \( \vec{n} = (1, 1, -\frac{1}{2}) \) is just \(-\frac{1}{2}\) times the previous, so of course gives same results!
The Gradient and Level Surfaces

For example, the function $F(x,y,z) = \frac{x^2}{1^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = \frac{1}{2}$

for $c$ some constant, $c > 0$ are a family of concentric ellipsoids.

All points $\mathbf{p} = (x,y,z)$ which satisfy $F(\mathbf{p}) = c$, for a fixed value of $c$

constitute a level surface for $F$.

When $c = 1$ we have the standard ellipsoid which extends from -1 to 1 along the x-axis, from -2 to 2 along the y-axis and from -3 to 3 along the z-axis.

For $c > 1$, the level surface is a larger but concentric ellipsoid and for $c < 1$, the level surface is a smaller but concentric ellipsoid.

Taking the gradient of $F(x)$, one obtains

\[
\nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = \left( 2x, \frac{y}{2}, \frac{2z}{9} \right)
\]

If we consider the point $\mathbf{p} = (\frac{1}{2}, 2, \frac{9}{2})$

it lies on the level surface for which $c = 3.5$

and we have $\nabla F(p) = (1, 1, 1)$

Thus at $\mathbf{p}$, the maximum rate of increase of $F$ is

\[
\| \nabla F(\mathbf{p}) \| = \|(1, 1, 1)\| = \sqrt{3}
\]
The direction of the maximum rate of increase at \( \vec{p} \) is the direction \((1, 1, 1)\) which as a unit direction is
\[
\vec{n} = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
\]
In the opposite direction, i.e., \(-\vec{n}\), the function \( F \) has its maximum rate of decrease, of value \(-\sqrt{3}\). At \( \vec{p} \), going in any direction perpendicular to \( \vec{n} \), the function is locally not changing. That is, directions perpendicular to \( \vec{n} \) all lie in a plane which is tangent to the surface.
Let's evaluate \( F(x, y, z) \) at points close to \( \vec{p} \).
First change only \( x \) by \( \Delta x = 0.01 \)
\[
F \left( \frac{1}{2} + 0.01, \frac{1}{2}, \frac{9}{2} \right) = 3.510100
\]
Thus \( \frac{\Delta F}{\Delta x} = \frac{3.510100 - 3.5}{0.01} = 1.0100 \)
Now change only \( y \) by \( \Delta y = 0.01 \)
This gives \( F \left( \frac{1}{2}, \frac{1}{2} + 0.01, \frac{9}{2} \right) = 3.510025 \)
\[
\frac{\Delta F}{\Delta y} = 1.0025
\]
And then change only \( z \) by \( \Delta z = 0.01 \)
Then \( F \left( \frac{1}{2}, \frac{1}{2}, \frac{9}{2} + 0.01 \right) = 3.510011 \)
\[
\frac{\Delta F}{\Delta z} = 1.0011
\]
These are consistent with \( \nabla F(\vec{p}) = (1, 1, 1) \) since the gradient is for the limit as \( \Delta \vec{s} \to 0 \)
Now evaluate \( F(\vec{x}) \) at
\[
\vec{p} + \Delta \vec{s} = \vec{p} + 0.01 \vec{n}, \text{ i.e., change of } 0.01
\]
in the direction of the maximum rate of increase
\[
F \left( \frac{1}{2} + \frac{0.01}{\sqrt{3}}, \frac{1}{2} + \frac{0.01}{\sqrt{3}}, \frac{9}{2} + \frac{0.01}{\sqrt{3}} \right) = 3.517366
\]
Now \( \frac{\Delta F}{\|\Delta \vec{s}\|} = \frac{3.517366 - 3.5}{0.01} = 1.7366 \) COMPARE TO \( \sqrt{3} \approx 1.7321 \)