Antiderivative

A function \( F(x) \) is an antiderivative of \( f(x) \) on an interval \( I \) if \( \frac{dF(x)}{dx} = f(x) \) for all \( x \) in \( I \).

If \( F(x) \) is an antiderivative of \( f(x) \) on an interval \( I \), then the most general antiderivative of \( f(x) \) on \( I \) is

\[ F(x) + C \]

where \( C \) is any constant.

Notation: The antiderivative of \( f(x) \) is denoted \( \int f(x) \, dx \)
It is also called an indefinite integral of \( f(x) \). We will soon be using indefinite integrals, i.e. antiderivatives, as the first step in evaluating a definite integral.

If you think that \( F(x) \) is the antiderivative of \( f(x) \), i.e. that \( F(x) = \int f(x) \, dx \), then just take \( \frac{dF(x)}{dx} \) and if you obtain \( f(x) \) then you have correctly found a particular antiderivative of \( f(x) \). Just add \( C \), i.e. \( F(x) + C \), and you have the general antiderivative of \( f(x) \). Then think about the interval \( I \). Sometimes you need to be concerned but often not.
Every derivative result you know lets you say something about antiderivatives.

1. \( \frac{d}{dx} (cf(x)) = c \frac{df(x)}{dx} \) tells you \( \int cf(x) \, dx = c \int f(x) \, dx \)

2. \( \frac{d}{dx} (f(x) \pm g(x)) = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx} \)
   tells you \( \int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \)

3. \( \frac{d}{dx} (x^n) = nx^{n-1} \) tells you \( \int x^n \, dx = \frac{x^{n+1}}{n+1} \), \( n \neq -1 \)

4. \( \frac{d}{dx} (\ln |x|) = \frac{1}{|x|} \frac{1}{x} = \frac{1}{x} \) tells you
   \( \int x^{-1} \, dx = \int \frac{dx}{x} = \ln |x| \), \( x \neq 0 \)

   (here the interval \( I \) can not include \( 0 \).
   The result is true for \(-\infty < x < 0\) and for \( 0 < x < \infty \)

5. \( \frac{d}{dx} (e^x) = e^x \) tells you \( \int e^x \, dx = e^x \)

Remember, these, and results to follow are particular antiderivatives. Adding a constant of integration makes them general antiderivatives.

What does the chain rule tell us? The chain rule for taking a derivative says that if
\( h(x) = f(g(x)) \) then
\( \frac{dh(x)}{dx} = f'(g(x)) g'(x) \)

For example, if \( h(x) = (kx+b)^3 \)
then \( \frac{dh(x)}{dx} = 3(kx+b)^2 \frac{d}{dx}(kx+b) = 3(kx+b)^2 k \)
For now, (like in the example), we will focus on a \( g(x) \) function that is linear, i.e.
\[ g(x) = kx + b \], \( k \) and \( b \) constants. The chain rule tells us that the derivative of a \( f(g(x)) \) for such a \( g(x) \) is just the derivative of the \( f \) function, still with argument \( g(x) \), but then multiplied by \( k \) which is the derivative of the \( g \) function. Thus the anti-derivative results 3, 4, and 5 generalize to:

3. \[ \int (kx+b)^n \, dx = \left( \frac{1}{k} \right) \left( \frac{(kx+b)^{n+1}}{n+1} \right) , \quad n \neq -1, \quad k \neq 0 \]

4. \[ \int \frac{dx}{kx+b} = \int (kx+b)^{-1} \, dx = \left( \frac{1}{k} \right) \ln |kx+b| , \quad k \neq 0 \]

5. \[ \int e^{kx+b} \, dx = \left( \frac{1}{k} \right) e^{kx+b} , \quad k \neq 0 \]

Notice that if the \( k \) is zero we don't have a function involving \( x \), but just \( \int adx = ax + c \), a some constant.

The case \( \int dx = x \) may confuse you at first.

Let's continue with more that you already "know":

6. \[ \frac{d}{dx} (\sin x) = \cos x \] tells you \( \int \cos x \, dx = \sin x \)

7. \[ \frac{d}{dx} (\cos x) = -\sin x \] tells you \( \int \sin x \, dx = -\cos x \)

Be careful for the sign of these two!

8. \[ \frac{d}{dx} (\tan x) = \sec^2 x \] tells you \( \int \sec^2 x \, dx = \tan x \)

9. \[ \frac{d}{dx} (\cot x) = -\csc^2 x \] tells you \( \int \csc^2 x \, dx = -\cot x \)
10. $\frac{d}{dx}(\sec x) = \sec x \tan x$ tells you $\int \sec x \tan x \, dx = \sec x$

11. $\frac{d}{dx}(\csc x) = -\csc x \cot x$ tells you $\int \csc x \cot x \, dx = -\csc x$

12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ tells you $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$

But don’t waste your time on this, learn a more general $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$

13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ tells you $\int \frac{dx}{1+x^2} = \tan^{-1} x$

But instead learn $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

Notice we do not have an "easy" result for standards such as $\int \ln x \, dx$, $\int \sec x \, dx$, $\int \csc x \, dx$, $\int \tan x \, dx$ and $\int \cot x \, dx$.

So these results are just an easy beginning.

Many, many functions which are called elementary functions are easy to differentiate but have antiderivatives which are not representable using elementary functions. Example $\int \sin(x^2) \, dx$

Problems for students: Find the general antiderivatives

1. $\int (3x^2 - 2x + 4) \, dx =$

2. $\int \left( x^{3/2} + 2e^{2x} \right) \, dx =$

3. $\int 2 \sin \left( \pi x + \frac{\pi}{2} \right) \, dx =$

4. $\int \frac{dx}{3x-2} =$