Derivation of the dot product

1. Notation:
   We will express 3 dimensional vectors as ordered triples:
   \[ \vec{a} = (a_x, a_y, a_z) \]

   This is equivalent to:
   \[ \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \]

2. Definitions:
   i. The magnitude of the vector \( \vec{a} \) written above is defined to be:
      \[ |\vec{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2} \]
      This definition follows from the Pythagorean Theorem.

   ii. The dot product of two vectors \( \vec{a} \) and \( \vec{b} \) is defined to be:
      \[ \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos \phi \]
      Where \( \phi \) is the angle between the two vectors.

3. We can use the law of cosines to determine the algebraic form of the dot product.
According to figure 1, the side of the triangle opposite $\phi$ is the vector $\vec{b} - \vec{a}$. From the rules of vector addition, this vector can be expressed as:

$$\vec{b} - \vec{a} = (b_x - a_x, b_y - a_y, b_z - a_z)$$

Its magnitude is thus:

$$|\vec{b} - \vec{a}| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2 + (b_z - a_z)^2}$$

Applying the law of cosines to this triangle yields the equation:

$$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2 \cdot |\vec{a}| \cdot |\vec{b}| \cos \phi$$

Substituting the definition of the dot product into this equation gives:

$$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})$$

We can now solve for $\vec{a} \cdot \vec{b}$.

$$\vec{a} \cdot \vec{b} = \frac{1}{2} \left( |\vec{a}|^2 + |\vec{b}|^2 - |\vec{b} - \vec{a}|^2 \right)$$

Now, let's put this in terms of the components of $\vec{a}$ and $\vec{b}$, then simplify:

$$\vec{a} \cdot \vec{b} = \frac{1}{2} \left( a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2 - (b_x - a_x)^2 - (b_y - a_y)^2 - (b_z - a_z)^2 \right)$$

A few steps of algebra will get us to the conclusion:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

We're done! We now know how to express the dot product in terms of the components of each vector! I hope you are as excited as I am about this.