1 Basic vector operations

Magnitude and unit vector

A vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ (a,b and c are scalars) can also be written as $\vec{v} = (a, b, c)$. The arrow in the symbol $\vec{v}$ indicates that it is a vector, which is a quantity that has a magnitude and a direction. The magnitude of $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, represented by $|\vec{v}|$ can be found by the following formula:

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$$

A unit vector is a vector with magnitude equal to 1. The unit vector in the direction of $\vec{v}$, represented by $\hat{v}$, can be found by the following formula:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

Dot product

The dot product is also known as scalar product. The result of a scalar product is a scalar. One can compute the scalar product of two 3-component vectors by summing up the products of the corresponding components in the two vectors. Here is an example. Given two vectors $\vec{v}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$. Their dot product would be:

$$\vec{v}_1 \cdot \vec{v}_2 = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

Here are some more facts about dot product that you should remember.

1. $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_2 \cdot \vec{v}_1$

2. The dot product can also be computed by:

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos \theta$$

where $\theta$ is the angle between the vectors $\vec{v}_1$ and $\vec{v}_2$. Conversely, if you know the magnitudes and dot product of the two vectors $\vec{v}_1$ and $\vec{v}_2$, you can use this formula to find out the angle $\theta$ between the two vectors.
3. The magnitude of a vector \( \vec{v} \) can be computed from its dot product with itself, according to the following formula:

\[
|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}
\]

4. Any two vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) that are perpendicular to one another have:

\[
\vec{v}_1 \cdot \vec{v}_2 = 0
\]

**Cross product**

The cross product is also known as vector product. The result of a vector product is a vector. One can compute the vector product of two 3-component vectors by setting up a 3-by-3 determinant. First, put the unit vectors \( \hat{i}, \hat{j} \) and \( \hat{k} \) in the first row. Then, put each of the 3 components of the first vector in the first row, and each components of the second vector in the second row. Expanding the determinant gives the cross product. Here is an example. Given two vectors \( \vec{v}_1 = \hat{i} + 2\hat{j} + 3\hat{k} \) and \( \vec{v}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k} \). Their cross product would be:

\[
\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 2 & 3 \\
4 & 5 & 6
\end{vmatrix} = \hat{i} \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3\hat{i} + 6\hat{j} - 3\hat{k}
\]

Here are some more facts about dot product that you should remember.

1. \( \vec{v}_1 \times \vec{v}_2 = -\vec{v}_2 \times \vec{v}_1 \) (not equal to \( +\vec{v}_2 \times \vec{v}_1 \) in general)

2. The magnitude of the cross product can be computed by:

\[
|\vec{v}_1 \times \vec{v}_2| = |\vec{v}_1||\vec{v}_2|\sin\theta
\]

where \( \theta \) is the angle between the vectors \( \vec{v}_1 \) and \( \vec{v}_2 \).

3. The vector product \( \vec{v}_1 \times \vec{v}_2 \) of two vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) is always perpendicular to both \( \vec{v}_1 \) and \( \vec{v}_2 \). In other words,

\[
(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_1 = (\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_2 = 0
\]

**Practice problems**

1. Given \( \vec{v} = \hat{i} + 9\hat{j} + 2\hat{k} \). Find \( |\vec{v}| \). (Answer: \( \sqrt{86} \))

2. Given \( \vec{w} = 3\hat{i} + 6\hat{j} + 9\hat{k} \). Find \( \hat{w} \), the unit vector in the direction of \( \vec{w} \). (Answer: \( \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k} \))

3. Given \( \vec{u} = 2\hat{i} + 10\hat{j} + \hat{k} \), \( \vec{v} = 15\hat{i} + 20\hat{j} \). Find \( \vec{u} \cdot \vec{v} \). (Answer: 230)

4. Given \( \vec{x} = (8, 9, 12) \), \( \vec{y} = (2, 3, 1) \). Find \( \vec{x} \cdot \vec{y} \) and \( \vec{y} \cdot \vec{x} \). (Answer: 55, 55)

5. Given \( \vec{v}_1 = 5\hat{i} + 5\hat{j} + 6\hat{k} \), \( \vec{v}_2 = 10\hat{i} + 20\hat{j} + 15\hat{k} \). Find \( \vec{v}_1 \times \vec{v}_2 \). (Answer: \( -45\hat{i} - 15\hat{j} + 50\hat{k} \))

6. Given \( \vec{u}_1 = (1, 3, 2) \), \( \vec{v}_2 = (7, 8, 5) \). Find \( \vec{u}_1 \times \vec{u}_2 \) and \( \vec{u}_2 \times \vec{u}_1 \). (Answer: \( (16, 9, -13), (-16, -9, 13) \))
2 Linear Dependence

Criterion for linear independence

The criterion for \( n \) vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) to be \emph{linearly independent} is that for scalars \( c_1, c_2, \ldots, c_n \),
\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0
\]

Here is an example. Consider the vectors \( \vec{v}_1 = 3\hat{i}, \vec{v}_2 = \hat{i} + 4\hat{j} + 8\hat{k} \) and \( \vec{v}_3 = 2\hat{i} + 9\hat{j} \). Let \( c_1, c_2, c_3 \) be three scalars. Now, we check the criterion for linear independence:
\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = 0 \Rightarrow c_1(3\hat{i}) + c_2(\hat{i} + 4\hat{j} + 8\hat{k}) + c_3(2\hat{i} + 9\hat{j}) = 0
\]
\[
\Rightarrow (3c_1 + c_2 + 2c_3)\hat{i} + (4c_2 + 9c_3)\hat{j} + 8c_2\hat{k} = 0
\]
\[
\Rightarrow 3c_1 + c_2 + 2c_3 = 0 \text{ and } 4c_2 + 9c_3 = 0 \text{ and } 8c_2 = 0
\]
\[
\Rightarrow c_1 = c_2 = c_3 = 0
\]

Since the criterion is satisfied, we can now conclude that \( \vec{v}_1 = 3\hat{i}, \vec{v}_2 = \hat{i} + 4\hat{j} + 8\hat{k} \) and \( \vec{v}_3 = 2\hat{i} + 9\hat{j} \) are linearly independent.

Test for linear independence using determinant

We can also check whether vectors are linearly independent by computing a determinant whose rows are the vectors to be tested. If the determinant turns out to be non-zero, then the vectors are linearly independent. As an example, for the vectors \( \vec{v}_1 = 3\hat{i}, \vec{v}_2 = \hat{i} + 4\hat{j} + 8\hat{k} \) and \( \vec{v}_3 = 2\hat{i} + 9\hat{j} \), we can check if they are linearly independent by evaluating the following determinant:
\[
\begin{vmatrix}
3 & 0 & 0 \\
1 & 4 & 8 \\
2 & 9 & 0 \\
\end{vmatrix}
\]

The value of this determinant turns out to be -216, which is not equal to zero. Therefore, we can conclude that the vectors \( \vec{v}_1 = 3\hat{i}, \vec{v}_2 = \hat{i} + 4\hat{j} + 8\hat{k} \) and \( \vec{v}_3 = 2\hat{i} + 9\hat{j} \) are linearly independent. However, it should be noted that this method only works when the number of vectors and the number of components in each vectors are equal. Otherwise, say, if you include an addition vector \( \vec{v}_4 = \hat{i} + 6\hat{j} + 5\hat{k} \), the determinant above will become a 4-by-3 one, which is undefined.

Practice problems

Check whether the following sets of vectors are linearly independent:

1. \( \vec{v}_1 = \hat{i}, \vec{v}_2 = 2\hat{i} + 8\hat{j}, \vec{v}_3 = 30\hat{k} \) (Answer: linearly independent)
2. \( \vec{v}_1 = 2\hat{i}, \vec{v}_2 = 3\hat{j}, \vec{v}_3 = 5\hat{i} + \hat{k} \) (Answer: linearly independent)
3. \( \vec{x} = (1, 2, 5), \vec{y} = (5, 0, 6), \vec{z} = (3, 4, 2) \) (Answer: linearly dependent)
4. \( \vec{a} = (8, 10, 90), \vec{b} = (120, 56, 98), \vec{c} = (0, 0, 0), \vec{d} = (-8, 0, 300) \) (Answer: linearly dependent)
5. \( \vec{u} = (5, 6, 9, 0), \vec{v} = (2, 0, 3, 4), \vec{w} = (1, 0, 0, 0) \) (Answer: linearly independent)
3 Limits of Multivariable Functions

Examples

The idea is best illustrated with examples. First, let us take a look at a relatively straight forward one. Here we want to evaluate: \( \lim_{x \to 0, y \to 0} e^{x^2} \sin(xy^2) \). The way to do this is by replacing the x’s and y’s in the function \( e^{x^2} \sin(xy^2) \) by zero, and then simplify.

\[
\lim_{x \to 0, y \to 0} e^{x^2} \sin(xy^2) = e^0 \sin(0) = 1 \cdot 0 = 0
\]

Now, let’s take a look at another example. This time we want to evaluate:

\[
\lim_{x \to \infty, y \to \infty} \frac{1 + 8x^3y^2}{2x^5y^6 + 3y}
\]

The function looks very complicate at the first glance. However, the problem would become much easier if you divide both the numerator and the denominator by the highest powers of x and y that you see in the function.

\[
\lim_{x \to \infty, y \to \infty} \frac{1 + 8x^3y^2}{2x^5y^6 + 3y} = \lim_{x \to \infty, y \to \infty} \frac{(1/x^5y^6)(1 + 8x^3y^2)}{(1/x^5y^6)(2x^6y^6 + 3y)} = \lim_{x \to \infty, y \to \infty} \frac{1/x^5y^6 + 8/x^2y^4}{2 + 3/x^5y^3} = 0 + 0 = 0
\]

It should be noted that limit does not always exist. Here is an example in which the limit does not exist:

\[
\lim_{x \to 0, y \to 0} \frac{\ln(1 + xy)}{\sin(xy^3)}
\]

In this case, both the numerator and the denominator tend to zero. Limit does not exist.

Practice problems

Evaluate the following limits, if any:

1. \( \lim_{x \to 0, y \to 0} e^{x^2+9xy} \) (Answer: 1)
2. \( \lim_{x \to 0, y \to 0} x^2e^{1+2x+3y+2x^3y} \) (Answer: 0)
3. \( \lim_{x \to \infty, y \to \infty} xy^3e^{-x} \) (Answer: limit does not exist)
4. \( \lim_{x \to \infty, y \to \infty} \frac{10x^3y^2+x+100y}{5x^3y^2+30x^2y^2+1000x} \) (Answer: 2)
5. \( \lim_{x \to 0, y \to 0} \frac{x}{x^3+y^3} \) (Answer: limit does not exist)
6. \( \lim_{x \to 0, y \to 0} \frac{(1+x) \sin y}{y} \)

Solution: \( \lim_{x \to 0, y \to 0} \frac{(1+x) \sin y}{y} = \lim_{x \to 0, y \to 0} (1 + x) \cdot \lim_{x \to 0, y \to 0} \frac{\sin y}{y} = \lim_{x \to 0} (1 + x) \cdot \lim_{y \to 0} \frac{\sin y}{y} = (1 + 0) \cdot 0 = 0 \)
4 Finding Partial Derivatives

Techniques

Let us consider a function $f$ of two variables $x$ and $y$, defined as $f(x, y) = x^2y^3 + x\sin y$. The partial derivative $\frac{\partial f}{\partial x}$ can be found by differentiating $f$ with respect to $x$ and treating $y$ as a constant. Therefore,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3 + \sin y) = 2xy^3 + \sin y$$

Similarly, the partial derivative $\frac{\partial f}{\partial y}$ can be found by differentiating $f$ with respect to $y$ and treating $x$ as a constant.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 y + \sin y) = 3x^2y^2 + \cos y$$

Now we know the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. We can then find out the total differential $df$ of the function $f$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (2xy^3 + \sin y)dx + (3x^2y^2 + \cos y)dy$$

The partial derivatives can also be used to find out the Jacobian Matrix, which is defined as in equation (2.32) on page 91 of the textbook by Wilfred Kaplan.

Practice problems

1. Given $f(x, y) = 2x + 3y - \cos(y)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. (Answer: 2, 3 + \sin(y))

2. Given $f(x, y) = x^2e^x \sin(x) + 2x^2y^6$. Find $\frac{\partial f}{\partial y}$. (Answer: 12x^2y^5)

3. Given $f(x, y) = e^x \tan y$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. (Answer: $e^x \tan y, e^x \sec^2 y$)

4. Pratice on chain rule: Given $f(x, y) = \ln(xy^3)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution: 
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \ln(xy^3) = \frac{1}{xy^3} \cdot y^3 = \frac{1}{x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \ln(xy^3) = \frac{1}{xy^3} \cdot 3xy^2 = \frac{3}{y}$$

5. Practice on product rule: Given $f(x, y) = e^{xy} \sin(xy)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Hence write down the total differential $df$.

Solution: 
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} e^{xy} \sin(xy)$$
$$= e^{xy} \cdot \cos(xy) \cdot y + \sin(xy) \cdot e^{xy} \cdot y$$
$$= ye^{xy}(\sin(xy) + \cos(xy))$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} e^{xy} \sin(xy)$$
$$= e^{xy} \cdot \cos(xy) \cdot x + \sin(xy) \cdot e^{xy} \cdot x$$
$$= xe^{xy}(\sin(xy) + \cos(xy))$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
$$= (ye^{xy}(\sin(xy) + \cos(xy)))dx + (xe^{xy}(\sin(xy) + \cos(xy)))dy$$
$$= e^{xy}(\sin(xy) + \cos(xy))(ydx + xdy)$$
5 Partial Derivatives of Implicit Functions

Example 1
Consider the implicit function \( z(x, y) \) defined by \( x^2 = z \sin(yz) \). Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

We start from the given equation that relates \( x, y \) and \( z \):

\[
x^2 = z \sin(yz)
\]

Now, we take the differential on both sides. Remember to apply product rule on the right hand side.

\[
2x \, dx = \sin(yz) \, dz + z \cos(yz) \left( y \, dz + z \, dy \right)
\]

\[
\Rightarrow \quad 2x \, dx = (\sin(yz) + yz \cos(yz)) \, dz + z^2 \cos(yz) \, dy
\]

\[
\Rightarrow \quad 2x \, dx - z^2 \cos(yz) \, dy = (\sin(yz) + yz \cos(yz)) \, dz
\]

\[
\Rightarrow \quad dz = \frac{2x \, dx - z^2 \cos(yz) \, dy}{\left( \sin(yz) + yz \cos(yz) \right)}
\]

\[
\Rightarrow \quad dz = \left( \frac{2x}{\left( \sin(yz) + yz \cos(yz) \right)} \right) \, dx - \frac{z^2 \cos(yz)}{\left( \sin(yz) + yz \cos(yz) \right)} \, dy
\]

Finally, read off the partial derivatives from the expression for the differential of \( z \). This gives:

\[
\frac{\partial z}{\partial x} = \frac{2x}{\left( \sin(yz) + yz \cos(yz) \right)} \quad \frac{\partial z}{\partial y} = -\frac{z^2 \cos(yz)}{\left( \sin(yz) + yz \cos(yz) \right)}
\]

Example 2
Consider the implicit functions \( y_1(x_1, x_2) \) and \( y_2(x_1, x_2) \) defined by:

\[
2x_1^2 + x_2^2 + y_1^2 - y_1 y_2 = 1
\]

\[
x_1^2 + x_2^2 + 2y_1^2 + y_1 y_2 = 3
\]

First, we move all the terms in each of the two equations to the left side, leaving the right side zero, and then define the left sides of the two equations to be two functions, \( F_1 \) and \( F_2 \), respectively.

\[
F_1(x_1, x_2, y_1(x_1, x_2), y_2(x_1, x_2)) = 2x_1^2 + x_2^2 + y_1^2 - y_1 y_2 - 1
\]

\[
F_2(x_1, x_2, y_1(x_1, x_2), y_2(x_1, x_2)) = x_1^2 + x_2^2 + 2y_1^2 + y_1 y_2 - 3
\]

Then, by the first formula in (2.61) on page 108 of the textbook by Wilfred Kaplan,

\[
\frac{\partial y_1}{\partial x_1} = \frac{\partial (F_1, F_2)}{\partial (x_1, y_2)} = \begin{vmatrix}
4x_1 & -y_1 \\
2x_1 & y_1
\end{vmatrix} = \begin{vmatrix}
2y_1 - y_2 & -y_1 \\
4y_1 + y_2 & y_1
\end{vmatrix} = \frac{-6x_1 y_1}{6y_1^2} = -\frac{x_1}{y_1}
\]

Practice Problems (on the next page)
1. An implicit function \( z(x, y) \) is defined by the equation \( x^2 \sin z = yz^2 \). Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \). 
(Answers: \( \frac{2x \sin z}{2yz - x^2 \cos z} \), \( \frac{-z^2}{2yz - x^2 \cos z} \))

2. An implicit function \( z(x, y) \) is defined by the equation \( xz^3 = y \tan z \). Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).
(Answers: \( \frac{-z^3}{3xz^2 - y \sec^2 z} \), \( \frac{\tan z}{3xz^2 - y \sec^2 z} \))

3. Consider the same implicit functions in the example 2. Use the other three formulae in (2.61) on page 108 of the textbook by Wilfred Kaplan,

\[
\frac{\partial y_1}{\partial x_2} = \frac{\partial (F_1, F_2)}{\partial (x_2, y_2)}(x_1, y_1), \quad \frac{\partial y_2}{\partial x_1} = \frac{\partial (F_1, F_2)}{\partial (x_1, y_1)}(y_1, y_2), \quad \frac{\partial y_2}{\partial x_2} = \frac{\partial (F_1, F_2)}{\partial (y_1, y_2)}(y_1, y_2)
\]

to show that

\[
\frac{\partial y_1}{\partial x_2} = \frac{2x_2 y_1}{3x_1^2}, \quad \frac{\partial y_2}{\partial x_1} = \frac{2y_1 + y_2}{6x_1}, \quad \frac{\partial y_2}{\partial x_2} = \frac{2x_2(y_1 + y_2)}{3x_1^2}
\]

6 Partial Derivatives of Inverse Functions

Example
Consider the mapping

\[
x_1(u_1, u_2) = u_1^2 - u_2^2, \quad x_2(u_1, u_2) = 2u_1 u_2.
\]

Its Jacobian matrix is:

\[
[x_u] = \begin{bmatrix} 2u_1 & -2u_2 \\ 2u_1 & 2u_2 \end{bmatrix}
\]

According to page 120 of the textbook by Wilfred Kaplan, the Jacobian matrix of the inverse mapping is simply the inverse of the Jacobian matrix of the mapping. In other words,

\[
[x_u]^{-1} = \begin{bmatrix} 2u_1 & -2u_2 \\ 2u_1 & 2u_2 \end{bmatrix}
\]

Now, recall the rule for inverting a \( 2 \times 2 \) matrix, which you learned in previous Mathematics courses:

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

Using this rule, we have:

\[
[x_u] = \frac{1}{4u_1^2 + 4u_2^2} \begin{bmatrix} 2u_1 & 2u_2 \\ -2u_2 & 2u_1 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ -\frac{u_1}{2u_1^2 + 2u_2^2} & \frac{u_2}{2u_1^2 + 2u_2^2} \end{bmatrix}
\]

This is Jacobian matrix of the inverse mapping \( u_1(x_1, x_2), u_2(x_1, x_2) \). We can read off from the Jacobian matrix that:

\[
\frac{\partial u_1}{\partial x_1} = \frac{u_1}{2u_1^2 + 2u_2^2}, \quad \frac{\partial u_1}{\partial x_2} = \frac{u_2}{2u_1^2 + 2u_2^2}
\]

\[
\frac{\partial u_2}{\partial x_1} = -\frac{u_2}{2u_1^2 + 2u_2^2}, \quad \frac{\partial u_2}{\partial x_2} = \frac{u_1}{2u_1^2 + 2u_2^2}
\]
Practice Problem
Consider the mapping \( x_1(u_1, u_2) = u_1 u_2 \), \( x_2(u_1, u_2) = u_1^3 \).
(a) Find the Jacobian matrix of this mapping, \([x_u]\).
(b) Find the Jacobian matrix of the inverse mapping, \([u_x]\).
(c) Find the partial derivatives of the inverse mapping, \( \frac{\partial u_i}{\partial x_j} \).
(Answers: \( \frac{\partial u_1}{\partial x_1} = 0, \frac{\partial u_1}{\partial x_2} = \frac{1}{3u_1^2}, \frac{\partial u_2}{\partial x_1} = \frac{1}{u_1}, \frac{\partial u_2}{\partial x_2} = -\frac{u_2}{3u_1^2} \))

7 Curvilinear Coordinates

Facts
Transforming from spherical coordinates to rectangular coordinates:
\[
\begin{align*}
    x &= \rho \sin \phi \cos \theta \\
    y &= \rho \sin \phi \sin \theta \\
    z &= \rho \cos \phi
\end{align*}
\]
Transforming from rectangular coordinates to spherical coordinates:
\[
\begin{align*}
    \rho &= \sqrt{x^2 + y^2 + z^2} \\
    \theta &= \arccos \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \arcsin \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \arctan \frac{y}{x} \\
    \phi &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \arctan \frac{\sqrt{x^2 + y^2}}{z}
\end{align*}
\]
Transforming from cylindrical coordinates to rectangular coordinates:
\[
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    z &= z
\end{align*}
\]
Transforming from rectangular coordinates to cylindrical coordinates:
\[
\begin{align*}
    r &= \sqrt{x^2 + y^2} \\
    \theta &= \arctan \frac{y}{x} \\
    z &= z
\end{align*}
\]

Practice Problems
1. Using the transformation from spherical coordinates to rectangular coordinates, show that
the Jacobian matrix is:
\[
\begin{pmatrix}
    \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\
    \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\
    \cos \phi & 0 & -\rho \sin \phi
\end{pmatrix}
\]
Hence, show that the Jacobian determinant is:
\[
\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 \sin \phi
\]
2. Using the transformation from cylindrical coordinates to rectangular coordinates, show that the Jacobian matrix is:
\[
\begin{pmatrix}
\cos \theta & -r \sin \theta & 0 \\
\sin \theta & r \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Hence, show that the Jacobian determinant is:
\[
\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r
\]

8 Gradient, Divergence, Curl and Laplacian

Definitions

\[\text{grad } f = \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}\]
\[\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\]
\[\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix}\]
\[\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\]

Practice Problems

1. Given \(f(x, y, z) = 3y^2 + x \sin z + \ln z\), find the gradient of \(f\).
   (Answer: \((\sin z)\hat{i} + 6y\hat{j} + (x \cos z + \frac{1}{z})\hat{k}\))

2. Given \(f(x, y, z) = 3 \exp(\tan(xyz))\), find the gradient of \(f\).
   (Answer: \(3 \exp(\tan(xyz)) \sec^2(xyz)(yz\hat{i} + xz\hat{j} + xy\hat{k})\))

3. Given \(f(r) = \frac{1}{r}\), find the gradient of \(f\). (Answer: \(-\frac{r}{r^2}\))

4. Given \(\vec{F}(x, y, z) = e^{xy}\hat{i} - e^{xy} \hat{j} + e^{yz} \hat{k}\), find the divergence of \(\vec{F}\).
   (Answer: \((y - x)e^{xy} + ye^{yz}\))

5. Given \(\vec{F}(x, y, z) = x\hat{i} + (y + \cos x)\hat{j} + (z + e^{xy})\hat{k}\), find the divergence of \(\vec{F}\). (Answer: 3)

6. Given \(\vec{F}(x, y, z) = x^3 \hat{i} + (x \sin(xy))\hat{j}\), find the divergence of \(\vec{F}\).
   (Answer: \((3 - \cos(xy))x^2\))

7. Given \(\vec{F}(x, y, z) = i \sin(xy) - j \cos(x^2y)\), find the divergence of \(\vec{F}\).
   (Answer: \(y \cos(xy) + x^2 \sin(x^2y)\))
8. Given \( \vec{F}(x, y, z) = (x^2 + y^2 + z^2)(3\hat{i} + 4\hat{j} + 5\hat{k}) \), find the curl of \( \vec{F} \).
   (Answer: \((10y - 8z)\hat{i} + (6z - 10x)\hat{j} + (8x - 6y)\hat{k}\))

9. Given \( \vec{F}(x, y, z) = (x + y)\hat{i} + (y + z)\hat{j} + (x + z)\hat{k} \), find the curl of \( \vec{F} \).
   (Answer: \(-\vec{r}\))

10. Given \( \vec{F}(x, y, z) = \hat{i} \sin x + \hat{j} \cos y \), find the scalar curl of \( \vec{F} \).
    (Answer: \(-\sin x\))

11. Given \( \vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k} \), show that \( \vec{F} \) is irrotational.

9 Geometrical Applications of the Gradient

Facts: Let \( S \) be a surface consisting of the points \((x, y, z)\) such that \( f(x, y, z) = 0 \).

1. The tangent plane to \( S \) at the point \((x_0, y_0, z_0)\) of \( S \) is given by
   \[ \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0 \]

2. The line normal to \( S \) at the point \((x_0, y_0, z_0)\) of \( S \) is given by
   \[ \vec{r}(t) = (x_0, y_0, z_0) + t\nabla f(x_0, y_0, z_0) \]

Practice Problems

Find the equation for the tangent plane to each of the following surfaces at the indicated point.

1. \( x^2 + y^2 + z^2 = 3 \), \((1, 1, 1)\)  \( \)(Answer: \( x + y + z = 3 \))

2. \( x^3 + z^3 = 2y^3 \), \((1, 1, 1)\)

3. \( e^z \cos x \cos y = 0 \), \((\frac{\pi}{2}, 1, 0)\)

4. \( e^{xyz} = 1 \), \((1, 2, 0)\)

Find the parametric equations for the normal line to each of the following surfaces at the indicated point.

1. \( x^2 + y^2 + 2z = 4 \), \((1, 1, 1)\)
   \( \)(Answer: \( x(t) = 1 + 2t, y(t) = 1 + 2t, z(t) = 1 + 2t \))

2. \( \tan(xy) = z \), \((\frac{\pi}{2}, \frac{\pi}{4}, 1)\)
10 Directional Derivatives

**Fact:** The directional derivative of a function $f$ along the direction $\vec{v}$ is found by

$$\nabla_{\vec{v}} f = \nabla f \cdot \hat{v} \quad with \quad \hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

**Practice Problems**

For each of the following functions $f(x, y)$, find the directional derivatives at the given point $(x_0, y_0)$ in the direction $\vec{v}$.

1. $f(x, y) = \cos(\sqrt{x^2 + y^2})$, $\vec{v} = \hat{i} + \hat{j}$, $(x_0, y_0) = (1, 1)$
   (Answer: $- \sin \sqrt{2}$)

2. $f(x, y) = \exp(-x^2 - y^2)$, $\vec{v} = \hat{i} + \hat{j}$, $(x_0, y_0) = (1, 1)$
   (Answer: $-2\sqrt{2} \exp(-2)$)

3. $f(x, y) = e^x + yz$, $\vec{v} = \hat{i} - \hat{j} + \hat{k}$, $(x_0, y_0) = (1, 1)$
   (Answer: $\frac{e}{\sqrt{3}}$)

4. $f(x, y) = xyz$, $\vec{v} = \hat{i} - \hat{k}$, $(x_0, y_0) = (1, 0)$
   (Answer: 0)

11 Critical Points of Multi-variable Functions

**Techniques**

If one wants to study about the critical points of a two-variable function $f(x, y)$, one should first locate the critical points. This can be achieved by setting

$$\frac{\partial f}{\partial x} = 0 \quad and \quad \frac{\partial f}{\partial y} = 0$$

and solve for the coordinates, $(x_0, y_0)$, of the critical points. For each of the critical points $(x_0, y_0)$, check the discriminant

$$D(x_0, y_0) = \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^2$$

1. If $D(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$, then $(x_0, y_0)$ is a local minimum.

2. If $D(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$, then $(x_0, y_0)$ is a local maximum.

3. If $D(x_0, y_0) < 0$, then $(x_0, y_0)$ is a saddle point.

**Practice Problems**

For each of the following functions, locate the critical point(s), and determine if each of them is a maximum, a minimum or a saddle point.
1. \( f(x, y) = \ln(x^2 + y^2 + 1) \) (Answer: Local minimum at \((0, 0)\))

2. \( f(x, y) = x^2 - y^2 + xy \) (Answer: Saddle point at \((0, 0)\))

3. \( f(x, y) = x^5y + xy^5 + xy \) (Answer: Saddle point at \((0, 0)\))

4. \( f(x, y) = \exp(1 + x^2 - y^2) \) (Answer: Saddle point at \((0, 0)\))

5. \( f(x, y) = 3x^2 + 2xy + 2x + y^2 + y + 4 \) (Answer: Local minimum at \((-\frac{1}{4}, -\frac{1}{4})\))

6. \( f(x, y) = xy + \frac{1}{x} + \frac{1}{y} \) (Answer: Local minimum at \((1, 1)\))

7. \( f(x, y) = x^3 + y^2 - 6xy + 6x + 3y \) (Answer: Saddle point at \((1, \frac{3}{2})\), Local minimum at \((5, \frac{47}{2})\))

8. \( f(x, y) = (x^2 + 3y^2) \exp(1 - x^2 - y^2) \) (Answer: Local minimum at \((0, 0)\), Local maxima at \((0, 1)\) and \((0, -1)\), Saddle points at \((1, 0)\) and \((-1, 0)\))

### 12 Lagrange Multipliers

**Techniques**

If we want to find the maximum and the minimum of a function \( f(x, y, z) \), subjected to a constraint \( g(x, y, z) = 0 \), we can use the Lagrange-multiplier method. This is done by setting up the following four equations, and solve for the coordinate \((x, y, z)\) of the extremum points.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\
\frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\
\frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} \\
g(x, y, z) &= 0
\end{align*}
\]

**Practice Problems**

In each of the following, find the extrema of the function \( f \) subjected to the stated constraint.

1. \( f(x, y, z) = x - y + z \), subjected to \( x^2 + y^2 + z^2 = 2 \) (Answer: Maximum at \((\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})\) and Minimum at \((-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}})\))

2. \( f(x, y) = x \), subjected to \( x^2 + 2y^2 = 3 \) (Answer: Maximum at \((\sqrt{3}, 0)\) and Minimum at \((-\sqrt{3}, 0)\))

3. \( f(x, y) = 3x + 2y \), subjected to \( 2x^2 + 3y^2 = 3 \) (Answer: Maximum at \((\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}})\) and Minimum at \((-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}})\))