Math 4B Integration Review (Solutions)
Do these integrals.
Solutions are posted at the website below.
If you have trouble with them, seek help immediately!

(1) \[ \int \frac{4}{\sqrt[3]{8x}} \cdot dx \]

(2) \[ \int \frac{2x + 1}{\sqrt{x^2 + x + 5}} \cdot dx \]

(3) \[ \int x \cdot e^{x^2} \cdot dx \]

(4) \[ \int x \cdot \sin \pi x \cdot dx \]

(5) \[ \int x^3 \cdot e^{2x} \cdot dx \]

(6) \[ \int e^x \cdot \cos x \cdot dx \]

(7) \[ \int \frac{x}{x^2 + 3x + 2} \cdot dx \]
We need to use a substitution to deal with the 8x, and then rewrite the root symbol as a fraction power.

This substitution always works the same way – if you have something like kx, you just end up dividing by k.

\[
(1) \int \frac{4}{\sqrt[3]{8x}} \cdot dx
\]

\[
u = 8x \Rightarrow du = 8dx \Rightarrow \frac{du}{8} = dx
\]

\[
\int \frac{4}{\sqrt[3]{u}} \cdot \frac{du}{8}
\]

\[
\frac{1}{2} \int u^{-\frac{1}{3}} \cdot du
\]

\[
\frac{1}{2} \left[ \frac{3}{2} u^{\frac{2}{3}} \right] + C
\]

\[
\frac{3}{4} (8x)^{\frac{2}{3}} + C
\]
(2) \[ \int \frac{2x + 1}{\sqrt{x^2 + x + 5}} \cdot dx \]

\[ u = x^2 + x + 5 \Rightarrow du = (2x + 1) \cdot dx \]

\[
\int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}} \cdot du
\]

\[ 2u^{\frac{1}{2}} + C \]

\[ 2\left(x^2 + x + 5\right)^{\frac{1}{2}} + C \]

Here you need to recognize that the derivative of the stuff in the square root is exactly what you have on top.

This makes the substitution work smoothly.
(3) \( \int x \cdot e^{x^2} \cdot dx \)

\[
u = x^2 \Rightarrow du = 2x \cdot dx \Rightarrow \frac{du}{2} = x \cdot dx
\]

\[
\int e^u \cdot \frac{du}{2}
\]

\[
\frac{1}{2} e^u + C
\]

Again, the derivative of the stuff in the exponential is multiplied in front, so the substitution will work nicely.

\[
\frac{1}{2} e^{x^2} + C
\]
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(4) \( \int x \cdot \sin \pi x \cdot dx \)

This one uses Integration by Parts

\[ \begin{align*}
    u &= x & dv &= \sin \pi x \cdot dx \\
    du &= dx & v &= -\frac{1}{\pi} \cos \pi x
\end{align*} \]

\[ \int u \cdot dv = uv - \int v \cdot du \]

\[ \begin{align*}
    \int x \cdot \sin \pi x \cdot dx &= -\frac{1}{\pi} x \cos \pi x - \int -\frac{1}{\pi} \cos \pi x \cdot dx \\
    &= -\frac{1}{\pi} x \cos \pi x + \frac{1}{\pi^2} \sin \pi x + C
\end{align*} \]

Be especially careful with negative signs in these type.
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(5) $\int x^3 \cdot e^{2x} \cdot dx$

We need integration by parts for this one, and we’ll need to do it 3 times (because of the $x^3$). For these repeated ones I like to use a shortcut so we don’t have to write down the intermediate solutions at each step.

Make a table with the u and the dv, then take derivatives of the u side until you get to zero, and integrate on the dv side. Then we can just write down the final solution, remembering to alternate signs.

Try this one the long way first, and prove to yourself why the shortcut works.

<table>
<thead>
<tr>
<th>u</th>
<th>dv</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3$</td>
<td>$e^{2x}$</td>
</tr>
<tr>
<td>$3x^2$</td>
<td>$\frac{1}{2} e^{2x}$</td>
</tr>
<tr>
<td>$6x$</td>
<td>$\frac{1}{4} e^{2x}$</td>
</tr>
<tr>
<td>$6$</td>
<td>$\frac{1}{8} e^{2x}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\frac{1}{16} e^{2x}$</td>
</tr>
</tbody>
</table>

This shortcut works because the function we use for the u part is a polynomial, which will go to zero eventually when we take enough derivatives. As long as the integrals are simple this trick saves you lots of time. It will work best for things like $\int x^n \cdot e^{ax} \cdot dx$ or $\int x^n \cdot \sin(ax) \cdot dx$

$$\int x^3 \cdot e^{2x} \cdot dx = \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} xe^{2x} - \frac{3}{8} e^{2x} + C$$
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(6) \( \int e^x \cdot \cos(x) \, dx \)

We use integration by parts twice for this one, but it’s a bit tricky. Make sure you write down both sides of the equation at each step – it will be much easier to remember what to do at the end.

\[
\int e^x \cdot \cos(x) \, dx = e^x \cdot \sin(x) - \int e^x \cdot \sin(x) \, dx
\]

After the first application of integration by parts it looks like we haven’t gotten anywhere, but we can expand the new integral using integration by parts again.

\[
\int e^x \cdot \cos(x) \, dx = e^x \cdot \sin(x) - \left[ -e^x \cdot \cos(x) - \int -e^x \cdot \cos(x) \, dx \right]
\]

\[
\int e^x \cdot \cos(x) \, dx = e^x \cdot \sin(x) + e^x \cdot \cos(x) - \int e^x \cdot \cos(x) \, dx
\]

2 \cdot \int e^x \cdot \cos(x) \, dx = e^x \cdot \sin(x) + e^x \cdot \cos(x)

\[
\int e^x \cdot \cos(x) \, dx = \frac{1}{2} \left[ e^x \cdot \sin(x) + e^x \cdot \cos(x) \right] + C
\]

we have come full circle to the integral we started with.

Add it to the other side and divide by 2 to get the final answer. Don’t forget to tack on the constant of integration.
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(7) \[ \int \frac{x}{x^2 + 3x + 2} \cdot dx \]

We could try a substitution trick like in integral (2), but it doesn’t quite work out.

Start by factoring the denominator:

\[ \int \frac{x}{(x + 1)(x + 2)} \cdot dx \]

We will use a partial fraction decomposition to turn this into 2 separate fractions, each of which will be easily integrated. This is just an algebra trick. Don’t panic.

\[
x = \frac{A}{x + 1} + \frac{B}{x + 2}
\]

\[
\frac{x}{(x + 1)(x + 2)} = \frac{A(x + 2) + B(x + 1)}{(x + 1)(x + 2)}
\]

\[
x = A(x + 2) + B(x + 1)
\]

\[
1x + 0 = (A + B)x + (2A + B)
\]

\[
1 = A + B \quad 0 = 2A + B
\]

\[
A = -1; B = 2
\]

\[
\frac{x}{(x + 1)(x + 2)} = \frac{-1}{x + 1} + \frac{2}{x + 2}
\]

Now we can do the integrals separately. In this case we get logarithms.

\[
\int \frac{x}{(x + 1)(x + 2)} \cdot dx
\]

\[
\int \frac{-1}{x + 1} + \frac{2}{x + 2} \cdot dx
\]

\[-1 \cdot \ln(x + 1) + 2 \cdot \ln(x + 2) + C\]