A Guided Tour of the Wave Equation

Background:

In order to solve this problem we need to review some facts about ordinary differential equations:

Some Common ODEs and their solutions:

\( f''(x) = 0 \quad \text{yields} \quad f(x) = ax + b \)

\( f''(x) + \omega^2 f(x) = 0 \quad \text{yields} \quad f(x) = A \cos(\omega x) + B \sin(\omega x) \)

\( f''(x) - \omega^2 f(x) = 0 \quad \text{yields} \quad f(x) = A e^{\omega x} + B e^{-\omega x} \)

You can verify that these solutions are correct by substituting them into the original differential equation, or by solving them directly by finding the roots of the characteristic polynomial.

Principle of Superposition:

If \( y_1 \) and \( y_2 \) are solutions to a linear homogeneous differential equation, then \( c_1 y_1 + c_2 y_2 \) is a solution to the same differential equation.

We are going to extend this principle to solutions of a linear homogeneous partial differential equation (of which the wave equation is one) and an infinite number of possible solutions:

If the functions \( u_1, u_2, u_3, \ldots \) are solutions to a linear homogeneous partial differential equation, then \( \sum_{n=1}^{\infty} b_n u_n \) is also a solution to the differential equation.

Functions of Different Variables:

Partial differential equations involve functions of more than one variable (in the case of the wave equation, the variables are \( x \) and \( t \)). The procedure we use will involve separating the solution into separate functions of the individual variables. The result will look something like this:

\[ F(x) = G(t) \]
These two functions are supposed to be equal to each other for all values of \( x \) and for all values of \( t \). The only way this can happen is if both functions are equal to some constant, which we shall call \( \lambda \).

So:

\[
F(x) = G(t) \quad \text{yields} \quad F(x) = \lambda \quad \text{and} \quad G(t) = \lambda
\]

**Fourier Sine Series:**

If a function \( h(x) \) is defined on the interval \([0, L]\), it can be written as an infinite series of the form:

\[
h(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)
\]

The coefficients can be found by the following formula:

\[
B_n = \frac{2}{L} \int_{0}^{L} h(x) \sin\left(\frac{n\pi x}{L}\right) dx
\]
The Problem:

A vibrating string with fixed endpoints:

Consider a string of length \( L \) vibrating back and forth in a single direction. The left end of the string is at \( x = 0 \), the right end of the string is at \( x = L \). The variable \( t \) denotes time. The function \( u(x,t) \) represents the displacement of the string from its equilibrium position at some position \( x \) along its length, and at some time \( t \).

The domain of \( u(x,t) \) is therefore: \( x \in [0, L], \ t \in [0, \infty) \)

The endpoints of the string are assumed to have displacement zero (since they are fixed.) This gives us some boundary conditions which we can use:

\[
\begin{align*}
u(0, t) &= 0, \\
u(L, t) &= 0
\end{align*}
\]

We can also assume we know the shape of the string initially (that is, the displacement of the string from equilibrium at every location when \( t \) is zero.)

\[
u(x, 0) = h(x)
\]

For this problem, we will assume the string is initially at rest. This means the transverse velocity of every point on the string is zero when \( t \) is zero.

\[
\frac{\partial u}{\partial t}(x, 0) = 0
\]

*If you like, you can solve this problem again when the initial transverse velocity is not zero.*

The differential equation that describes the motion of the string is:

\[
c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}
\]

Where \( c \) is a given constant.

Why this differential equation in particular describes waves on a string is a question to be answered on some other really long review sheet.
The Problem, Restated More Succinctly:

Find the function $u(x,t)$ with domain $x \in [0, L]$, $t \in [0, \infty)$ that satisfies the following:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$u(0, t) = u(L, t) = 0$

$u(x, 0) = h(x)$ for a given $h(x)$ with domain $x \in [0, L]$

$\frac{\partial u}{\partial t}(x, 0) = 0$

The Procedure: Separation of Variables

Assume $u(x,t)$ can be written as $u(x, t) = f(x)g(t)$

Note that it is not at all obvious that you can do this, but some mathematician over 200 years ago made this assumption and got a solution out of it, so it is a good assumption.

Substitute into the Differential Equation:

$$c^2 f''(x)g(t) = f(x)g''(t)$$

Now we can separate the variables. That is, get all the $x$’s on one side of the equation and all the $t$’s on the other. For no particular reason, I will group the $c^2$ with the $t$’s:

$$\frac{f''(x)}{f(x)} = \frac{g''(t)}{c^2 g(t)}$$

On the left hand side of this equation is a function of only $x$. On the right hand side is a function with only $t$. This means that the expression on the left and right must both be equal to some constant:

$$\frac{f''(x)}{f(x)} = \lambda \quad \text{and} \quad \frac{g''(t)}{c^2 g(t)} = \lambda$$

Since $\lambda$ can be any constant, each possible value will result in a different solution for $f(x)$ and $g(t)$. We will be interested in the ones that are non-zero.
Boundary Conditions:

The boundary conditions in the original problem become boundary conditions for \( f(x) \):

\[
\begin{align*}
    u(0, t) &= 0 \quad \text{yields} \quad f(0) = 0 \\
    u(L, t) &= 0 \quad \text{yields} \quad f(L) = 0
\end{align*}
\]

We can now solve the differential equation for \( f(x) \) for each value of \( \lambda \):

1. If \( \lambda \) is positive:

   Let \( \lambda = p^2 \) for some \( p > 0 \)

   The differential equation becomes: \( f''(x) = p^2 f(x) \)

   The general solution to this DE is: \( f(x) = Ae^{px} + Be^{-px} \)

   Plugging in the boundary conditions yields:

   \[
   \begin{align*}
   0 &= A + B \\
   0 &= Ae^{pL} + Be^{-pL}
   \end{align*}
   \]

   We can now solve for the constants \( A \) and \( B \): \( A = B = 0 \)

   **Conclusion:** if \( \lambda > 0 \), then \( f(x) = 0 \), and \( u(x, t) = 0 \)

2. If \( \lambda \) is zero:

   The differential equation becomes: \( f''(x) = 0 \)

   The general solution to this DE is: \( f(x) = ax + b \)

   Plugging in the boundary conditions yields:

   \[
   \begin{align*}
   0 &= b \\
   0 &= aL + b
   \end{align*}
   \]

   We can now solve for the constants \( a \) and \( b \): \( a = b = 0 \).

   **Conclusion:** if \( \lambda = 0 \), then \( f(x) = 0 \), and \( u(x, t) = 0 \)
3. **If $\lambda$ is negative:**

   Let $\lambda = -\omega^2$ for some $\omega > 0$

   The differential equation becomes: $f''(x) = -\omega^2 f(x)$

   The general solution to this DE is: $f(x) = A \cos(\omega x) + B \sin(\omega x)$

   Plug in the boundary conditions:

   $f(0) = 0$: $0 = A$

   This removes the cosine function from the general solution, and we are left with:

   $f(x) = B \sin(\omega x)$

   Now plug in the other boundary condition:

   $f(L) = 0$: $0 = B \sin(\omega L)$

   It is possible for the coefficient $B$ to be non-zero, but only for values of $\omega$ that make $\sin(\omega L)$ equal to zero. This happens if $\omega L = n\pi$ for some positive integer $n$.

   We therefore have two possibilities:

   **If $\lambda = -\left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \ldots$, then** $f(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$.

   **If $\lambda$ = anything else, then** $f(x) = 0$.

   We now know what $f(x)$ is for all values of $\lambda$. Now we need to find what $g(t)$ is, but only for the values of $\lambda$ where $f(x)$ can be non-zero.
Finding the function of time:

The zero initial condition:

One thing we can do to make this problem a little simpler is to deal with the second initial condition:

\[ \frac{\partial u}{\partial t}(x, 0) = 0. \]

This translates into condition for \( g(t) \):

\[ g'(0) = 0. \]

For each value of \( \lambda \) that yields a nonzero \( f(x) \), solve for \( g(t) \):

This differential equation becomes:

\[ g''(t) = -c^2 \left( \frac{n\pi}{L} \right)^2 g(t). \]

We can simplify the DE a bit so it looks more familiar:

\[ g''(t) = -\left( \frac{cn\pi}{L} \right)^2 g(t). \]

The general solution to this DE is:

\[ g(t) = A \cos \left( \frac{cn\pi t}{L} \right) + B \sin \left( \frac{cn\pi t}{L} \right). \]

We have one initial condition which we can use to simplify this function:

\[ g'(0) = 0 \quad \text{yields} \quad B = 0. \]

This means we have a \( g(t) \) for every important value of \( \lambda \), or equivalently, every value of \( n \):

\[ g(t) = A_n \cos \left( \frac{cn\pi t}{L} \right). \]

Putting it All Together:

What we have now is an infinite number of possible solutions to the partial differential equation, one for each value of \( n \).

\[ u_n(x, t) = B_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{cn\pi t}{L} \right). \]

*Note: The constants in front of each function merged together to make another constant.

Here’s where the principle of superposition comes in. Since the original PDE is linear and homogeneous, the sum of all the possible solutions is the general solution. So all we have to do is slap a sigma in front of the whole thing and we’ll have our general solution.
The General Solution:

\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{cn\pi t}{L} \right) \]

We’re almost done! The only thing left is to find the coefficients \( B_n \). We can do that by using the one piece of information we haven’t used yet, the initial condition:

\[ u(x, 0) = h(x) \]

Substituting this into the general solution yields:

\[ h(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \]

This means that \( u(x, 0) \) is the Fourier Sine Series for the function \( h(x) \). We can find the coefficients by computing the integral:

\[ B_n = \frac{2}{L} \int_{0}^{L} h(x) \sin \left( \frac{n\pi x}{L} \right) dx \]

Once the integral is computed and the coefficients are found, you may substitute the values of \( B_n \) into the general solution, and the partial differential equation will be completely solved!