Math 4A Midterm 1 Review Key

1. False, if \( \overrightarrow{u}_1 \) and \( \overrightarrow{u}_2 \) are a multiple of each other and/or if \( \overrightarrow{u}_1 = \overrightarrow{0} \) or \( \overrightarrow{u}_2 = \overrightarrow{0} \), then \( S \) would be linear dependent.

2. False, \( \mathbb{R}^3 \) is of dimension 3 and any set consisting of more than 3 vectors from \( \mathbb{R}^3 \) will always be linear dependent.

3. False, Since \( \mathbb{R}^3 \) is of dimension 3 we know that any basis set for \( \mathbb{R}^3 \) must have 3 vectors.

4. False, see answer for 3.

5. False, This is a set of two vectors only, if both \( \overrightarrow{u}_1 = \overrightarrow{u}_2 = \overrightarrow{0} \) then it would qualify as a vector subspace, But that is only way.

6. True, A span, i.e. all possible linear combinations, of a set of vectors is a vector subspace.

7. False, Both vector spaces and vector subspaces must satisfy the same 10 properties. However, sets chosen from a vector space already satisfy 7 of the properties and need to be tested only for 1. Include \( \overrightarrow{0} \). 2. Closed under vector addition 3. Closed under scalar multiplication, to determine if it is a vector subspace.

8. False, \( S \) will always be linear dependent, but \( \overrightarrow{u}_4 \) could be independent of the \( \overrightarrow{u}_1, \overrightarrow{u}_2, \) and \( \overrightarrow{u}_3 \).

9. Standard basis set \( \{ \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3 \} \) i.e. \( \{ [1, 0, 0], [0, 1, 0], [0, 0, 1] \} \)

10. \( \dim \mathbb{R}^3 = 3 \)
11. General polynomial (in \( P_2 \), \( p(x) = a_0 + a_1 x + a_2 x^2 \), \( a_0, a_1, a_2 \in \mathbb{R} \))

12. Yes, Test \( C_1 (1+t^2) + C_2 (1-t) = 0 = 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 \)

becomes \( \begin{align*}
(c_1 + c_2) \cdot 1 &= 0 \cdot 1 \\
-c_2 \cdot t &= 0 \cdot t \\
c_1 \cdot t^2 &= 0 \cdot t^2
\end{align*} \)

Only solution is \( c_1 = 0, c_2 = 0 \)

13. \( \text{No}, \ \dim P_2 = 3 \), set has only 2 vectors in it.

Clearly \( p(t) = 3 - t + t^2 \) is not in the span of \( S \)

14. One example add \( p(t) = 1 \) giving

\[ S_2 = \left\{ 1 + t^2, 1 - t, 1 \right\} \]

Then \( p(t) = a_0 + a_1 t + a_2 t^2 = a_2 (1 + t^2) - a_1 (1 - t) + (a_0 - a_1 - a_2) \cdot 1 \)

15. Standard basis set for \( P_2 = \left\{ 1, t, t^2 \right\} \)

or if want \( p(x) = \left\{ 1, x, x^2 \right\} \)

16. \( \dim P_2 = 3 \)

17. General vector in \( \mathbb{M}_{22} \) looks like \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( a, b, c, d \in \mathbb{R} \)

18. \( \dim \mathbb{M}_{22} = 4 \)

19. Standard basis set for \( \mathbb{M}_{22} \) is set \( \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \)

20. Since vectors in \( S \) are from a known vector space we need to only confirm that

- \( S \) includes the \( \overline{S} \) vector
- \( S \) is closed under vector addition, i.e., for \( \overline{u}, \overline{v} \in S \) the \( \overline{u} + \overline{v} \in S \).
- \( S \) is closed under scalar multiplication, i.e., for \( c \in \mathbb{R} \) and \( \overline{u} \in S \) the \( c \overline{u} \in S \).

21. See Answer to 7. Both satisfy the same 10 properties, but see Answer to 20, it is easier for you to check for subspace since need check only 3 of the properties
22. A general matrix (vector) in $M_{22}$ looks like $[ab,cd]$, $a,b,c,d \in \mathbb{R}$ but matrices in $W$ also have $b+c=0$ so $c=-b$, so look like $[a,b]$.

Yes

Check three requirements to be a vector subspace.

- Since $\overline{0} = [0,0]$, $\overline{0} \in W$

- Let $\overline{u} = [a_1,b_1]$, $\overline{v} = [a_2,b_2]$

then $\overline{u} + \overline{v} = [a_1+a_2, b_1+b_2]$ which is in $W$

- $k[a,b] = [ka, kb]$ which is in $W$

So $W$ is a vector subspace.

23. Yes. $\overline{0} = 0 + 0x + 0x^2$ satisfies $a_0 + a_1 = 0$

- Let $\overline{u} = a_0 + a_1x + a_2x^2$ with $a_0 + a_1 = 0$

$\overline{v} = b_0 + b_1x + b_2x^2$ with $b_0 + b_1 = 0$

$\overline{u} + \overline{v} = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$

but $(a_0+b_0) + (a_1+b_1) = 0$

since $a_0 + a_1 = 0$ and $b_0 + b_1 = 0$

- $c\overline{u} = c(a_0 + a_1x + a_2x^2) = ca_0 + ca_1x + ca_2x^2$

$ca_0 + ca_1 = c(a_0 + a_1) = 0$ since $a_0 + a_1 = 0$

24. No. Easy check. Does $W$ contain the zero polynomial, i.e., $p(x) = 0.1 + 0.1x + 0.1x^2 = 0$ so $a_0 = a_1 = a_2 = 0$. No, $a_0 + a_1 = 1$ is not satisfied.
25 Y, Apply test: Must have $T(\vec{a} + \vec{v}) = T(\vec{a}) + T(\vec{v})$

$T(c\vec{u}) = cT(\vec{u})$

Another easy check $T(\vec{0}) = \vec{0}$

$T(\vec{e}_1) = T(\vec{[1]}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $T(\vec{e}_2) = T(\vec{[0]}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Kernel, want $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for which $T(\vec{x}) = \vec{0}$

Thus $2x_1 + x_2 = 0$ Only solution

$-x_1 - 2x_2 = 0$ is $x_1 = 0, x_2 = 0$

So Kernel of $T = \{ \vec{0} \}$ (Single vector)

26 $T: \mathbb{R}^2 \to \mathbb{R}$ means we have a transformation with input vectors from $\mathbb{R}^2$ and output vectors (i.e. $T(\vec{x})$)

which are in $\mathbb{R}$.

Kernel of $T$ is all inputs $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

for which $T(\vec{x}) = \vec{0}$

i.e. $-x_1 + 2x_2 = 0$ i.e. $x_1 = 2x_2$

i.e any $c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Thus basis set for this Kernel $= \{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \}$

and Kernel = Span $\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \}$

i.e any $\vec{x} = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

i.e $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c \\ c \end{bmatrix}$

Check $T(\begin{bmatrix} 2c \\ c \end{bmatrix}) = \begin{bmatrix} -(2c) + 2(c) \end{bmatrix} = [0]$