Angles of a Triangle

Let's have some fun with

\[ \theta = \cos^{-1}\left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||} \right) \]

Find the value of the vertex angles of a triangle illustrated in \( \mathbb{R}^3 \):

\[ (2, 2, 5) \]
\[ (3, 4, -3) \]
\[ \mathbf{u} = (0, 0, 0) - (2, 2, 5) = (-2, -2, -5) \]
\[ \mathbf{v} = (3, 4, -3) - (2, 2, 5) = (1, 2, -8) \]

To evaluate \( \theta_1 \), I need \( \mathbf{u} \) and \( \mathbf{v} \) as shown.

Notice, both \( \mathbf{u} \) and \( \mathbf{v} \) are oriented with their tails at the \( \theta_1 \) vertex. Each vector is then its head coordinate less its tail coordinate.

\[ \mathbf{u} \cdot \mathbf{v} = -2 \cdot 4 + 40 = 34 \]
\[ ||\mathbf{u}|| = \sqrt{4 + 4 + 25} = \sqrt{33} \]
\[ ||\mathbf{v}|| = \sqrt{1 + 4 + 64} = \sqrt{69} \]
\[ \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||} = \frac{34}{\sqrt{33} \sqrt{69}} \approx 0.712521 \]
\[ \theta_1 = \cos^{-1}(0.712521) = 44.56^\circ \]

Students do \( \theta_2 \) and \( \theta_3 \).

Answer: \[ \theta_2 = \cos^{-1}\left( \frac{-1}{\sqrt{33} \sqrt{34}} \right) \approx 91.71^\circ \]

\[ \theta_3 = \cos^{-1}\left( \frac{35}{\sqrt{69} \sqrt{34}} \right) \approx 43.73^\circ \]