Solving Non-Homogeneous DE by Undetermined Coefficients

This method works only for linear, constant coefficient DE, with a forcing function (i.e., the part that makes it non-homogeneous) from a select set of functions.

The general solution of such a DE consists of the sum of the solution of the associated homogeneous DE and a unique particular solution of the non-homogeneous DE.

\[ y = y_h + y_p \]

For example, \( y'' - y' - 2y = t \) has the associated homogeneous DE.

\[ y'' - y' - 2y = 0 \]

\[ \lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda - 2)(\lambda + 1) = 0 \]

with solution

\[ y_h = c_1 e^{2t} + c_2 e^{-t} \]

The non-homogeneous DE has a particular solution

\[ y_p = -\frac{1}{2} + \frac{1}{4} \]

which you should confirm satisfies the DE.

The general solution is then

\[ y = y_h + y_p = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2} + \frac{1}{4} \]

Notice that the initial conditions, when
specified, allow the constants $c_1$ and $c_2$ in the $y_h$ portion to be determined. The allowed forcing functions are:
- Polynomials in $t$, such as $t^2-t+1$, $2t$, $4$, etc.
- Exponentials in $t$, such as $-2e^{-3t}$, $4e^{5t}$, etc.
- Sinusoids in $t$, such as $3\sin 2t$, $-2\cos 5t$, etc.
- Finite products and/or sums of the above.

To find the particular solution, assume a general function of the same form as the forcing function. In the example D.E. the forcing function $t$ is a member of $P_1$, so assume $y_p = At + B$, a general $P_1$.

Take derivatives as needed and plug into the D.E., equate common terms and solve for the $A$ and $B$ values.

So in this case: $y_p' = A, y_p'' = 0$
Plugging in gives:

$$0 - A - 2(At + B) = t$$
$$0 - A - 2At - 2B = t$$

Multiplies of $t$ $\rightarrow -2A = 1 \rightarrow A = -\frac{1}{2}$

Constants $\rightarrow -A - 2B = 0 \rightarrow B = -\frac{1}{2}A = \frac{1}{4}$

Thus $y_p = -\frac{1}{2}t + \frac{1}{4}$

And $y = y_h + y_p = c_1e^{2t} + c_2e^{-\frac{t}{2}} + \frac{1}{4}$

Now, and only now, choose $c_1$ and $c_2$ to satisfy any initial conditions.
Example choices for the particular solution:

<table>
<thead>
<tr>
<th>Forcing Function</th>
<th>Choice For Particular Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3t^2 + 2)</td>
<td>(At^2 + Bt + C)</td>
</tr>
<tr>
<td>(-4)</td>
<td>(A)</td>
</tr>
<tr>
<td>(2e^{-3t})</td>
<td>(Ae^{-3t})</td>
</tr>
<tr>
<td>(\sin(2t))</td>
<td>(A\cos(2t) + B\sin(2t))</td>
</tr>
<tr>
<td>(-\cos(3t) + \sin(t))</td>
<td>(A\cos(3t) + B\sin(3t) + C\cos(t) + D\sin(t))</td>
</tr>
<tr>
<td>(t^2e^{-t})</td>
<td>((At^2 + Bt + C)e^{-t})</td>
</tr>
<tr>
<td>(t\cos(t))</td>
<td>((At + B)\cos(t) + (Ct + D)\sin(t))</td>
</tr>
</tbody>
</table>

Note: Whenever a \(\sin\) or a \(\cos\) must have both \(\sin\) and \(\cos\) in the particular solution form.

**A MAJOR CAVEAT!** If any of the independent solution functions for the associated homogeneous D.E. is similar to the forcing function, the normal choice for the particular solution must be multiplied by \(t\) (possibly more than once) until there is no similarity.

Example: If one homogeneous solution function is \(t e^{-2t}\) and the forcing function is \(2te^{-2t}\) then the normal choice of particular solution would be

\[y_p = (At + B)e^{-2t}\]
Since the $At e^{-2t}$ part of this choice is similar to the homogeneous solution, $te^{-2t}$, we multiply by $t$ getting
\[ yp = (At^2 + Bt) e^{-2t} \]

However, now the $Bt e^{-2t}$ is similar to the homogeneous solution, thus requiring another multiplication by $t$. This results in the choice for particular solution being
\[ yp = (At^3 + Bt^2) e^{-2t} \]

Now there is no similarity, the $At^3 e^{-2t}$ and $Bt^2 e^{-2t}$ are both different from the homogeneous solution function $te^{-2t}$.

Because of this caveat, it is always wise to solve for the homogeneous solution functions first.

Note also, $e^{-2t}$, $te^{-2t}$, $e^{-2t}\sin(3t)$, $\sin(3t)$ and $te^{-2t}\sin(3t)$ are, for example, all independent and present no complication.