A system of linear equations is a bunch of linear equations:

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \]
\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \]

A solution to the system is a list of numbers \( x_1, \ldots, x_n \) so that all \( m \) equations are true.

Simplify the system until the set of solutions is obvious. Solutions may not exist, may be unique, may be infinite. Only three possibilities.

2) First step: All the info is in the numbers \( a_{ij}, b_i \)
So put them in a matrix:

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\]

\( m \times n \) coefficient matrix

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
  a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn} & b_m
\end{bmatrix}
\]

\( m \times (n + 1) \) augmented matrix
These operations don't change the set of solutions for the system:

- Reorder the rows (interchange) \( R_i \leftrightarrow R_j \)
- Multiply a row by \( c \neq 0 \) (scale) \( R_i^c = c R_i \)
- Add a multiple of one row to another row (replacement) \( R_i^* = R_i + k R_j \)

Grand strategy:
Do this until the equations are easy to solve.

Echelon form is pretty easy.
Reduced echelon form is easiest.

We will show how to get there and why this makes it easy.

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**Definition**

A matrix is in echelon form if
(1) all rows that are totally zero are at the bottom, and
(2) the first non-zero entry in any row, called the leading entry, is to the right of the leading entry of the previous row.

Example:

\[
\begin{bmatrix}
\$ & * & * & * & * & * \\
0 & \$ & * & * & * & * \\
0 & 0 & 0 & \$ & * & * \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\(\$ = \) non-zero entry (leading entry or pivot)
\(* = \) any entry
**Definition**

A matrix is in reduced echelon form if

1. it is in echelon form,
2. all leading entries are 1, and
3. in a column with a leading 1, all other entries are zero.

Example:

\[
\begin{bmatrix}
1 & 0 & * & 0 & * & * \\
0 & 1 & * & 0 & * & * \\
0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

**Theorem**

*Any matrix* $A$ *reduces to a unique matrix in reduced echelon form.*

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**6)**

Any matrix reduces to echelon form via these steps:

1. Ignore all totally zero columns on the left.
2. **Interchange** rows so that the top entry in the leftmost nonzero column is nonzero; call it the pivot.
3. **Use replacement** to zero out everything below the pivot.
4. Move on to the submatrix south-east of the pivot.
5. Repeat the procedure for this submatrix.
6. Repeat until done (i.e., nothing but zeroes to the southeast).

Echelon form changed to reduced row echelon (RREF) by:

1. **Scale** each non-zero row by dividing by its leading entry.
2. **Use replacement** to zero out everything above the pivots.
7) Why reduced echelon?

For a matrix in this form, answers to our question are obvious:

- If the last column of augmented matrix is pivot, there is no solution:

\[
\begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
\end{bmatrix} \Rightarrow 0x_1 + 0x_2 + \ldots 0x_n = 1, \text{ absurd}
\]

- If every column but the last is pivot, there is unique solution:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 2 \\
\end{bmatrix} \Rightarrow
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix}
1 \\
-3 \\
2 \\
\end{bmatrix}
\]

- If there are free variables, then infinitely many solutions.

Free variable(s) for column(s) without a pivot.

8) How to find all solutions when there are free variables:

- Set free variables to anything;

- Other variables are then completely determined.

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{3}{2} \\
0 & 1 & 0 & \frac{1}{2} \\
0 & 0 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \Rightarrow
\begin{bmatrix}
x_1 + x_2 = \frac{3}{2} \\
x_3 = 2 \\
x_4 = \frac{1}{2} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
\frac{3}{2} \\
0 \\
2 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
-1 \\
1 \\
0 \\
0 \\
\end{bmatrix} x_2, \text{ } x_2 \text{ any real number}
\]