The Fundamental Theorem of Calculus

**THE FUNDAMENTAL THEOREM OF CALCULUS, PART I** If \( f \) is continuous on \([a, b]\),
then the function \( g \) defined by

\[
g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b
\]

is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( g'(x) = f(x) \).

This is equivalent to:

\[
\frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x)
\]

*Simple example:*

If \( g(x) = \int_3^x e^{t^2-t} \, dt \)

\[
\frac{dg}{dx} = e^{x^2-x}
\]

*Complicated example:*

If \( g(x) = \int_{\sqrt{x}}^{x^3} \sqrt{t} \sin t \, dt \)

Rewrite \( g(x) \) as

\[
g(x) = \int_0^{\sqrt{x}} \sqrt{t} \sin t \, dt + \int_{\sqrt{x}}^{x^3} \sqrt{t} \sin t \, dt
\]

\[
= -\int_{\sqrt{x}}^{x^3} \sqrt{t} \sin t \, dt + \int_0^{\sqrt{x}} \sqrt{t} \sin t \, dt
\]

Then

\[
\frac{dg}{dx} = -\sqrt{x} \sin \sqrt{x} \frac{d}{dx} \left( \sqrt{x} \right) + \sqrt{x^3} \sin(x^3) \frac{d}{dx}(x^3)
\]

\[
= -\frac{\sqrt{x^3} \sin(x^3)}{2} + x^{3/2} \sin(x^3) \left( 3x^2 \right)
\]

\[
\frac{dg}{dx} = \frac{\sin \sqrt{x}}{2 \sqrt{x}} + 3x^{7/2} \sin(x^3)
\]
Note: the independent variable (here x) must be involved in only the upper limit and if the limit is a function of the independent variable the chain rule is needed.

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

where \( F \) is any antiderivative of \( f \), that is, a function such that \( F' = f \).

It is best to add another step:
\[
\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a)
\]

ie first write down an antiderivative function of \( f(x) \) and then evaluate that function at the upper limit and subtract the evaluation at the lower limit.

Simple example:
\[
\int_{-1}^{2} (x^3 - 2x) \, dx = \left( \frac{x^4}{4} - x^2 \right) \bigg|_{-1}^{2}
\]
\[
= \frac{2^4}{4} - 2^2 - \left( \frac{(-1)^4}{4} - (-1)^2 \right)
\]
\[
= \frac{16}{4} - 4 - \left( \frac{1}{4} - 1 \right)
\]
\[
= 4 - 4 - \frac{1}{4} + 1
\]
\[
= \frac{3}{4}
\]