Definite Integral and Riemann Sum

3. THEOREM If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x) \, dx$ exists.

4. THEOREM If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i \Delta x$$

The $\sum_{i=1}^{n} f(x_i) \Delta x$ is called a Riemann Sum and generalizes to $\sum_{i}^{n} f(x_i^*) \Delta x$ where $x_i^*$ is any point in the $i$th subinterval. It is generally convenient to use $x_i = a + i \Delta x$ giving $x_i$ as the right hand end point of the $i$th subinterval.

Some useful rules and results for sums are:

5. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7. $$\sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

The remaining formulas are simple rules for working with sigma notation:

8. $$\sum_{i=1}^{n} c = nc$$

9. $$\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$$

10. $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

11. $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$
Example: Find the expression for the Riemann sum of \( f(x) = x^2 + x \) integrated from \(-1\) to \(2\) using right-hand endpoints.

Write the Riemann sum for \( \int_{-1}^{2} (x^2 + x) \, dx \)

We have, \( a = -1 \), \( b = 2 \), so \( \Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n} \)

and then \( x_i = a + i \Delta x = -1 + i \frac{3}{n} \)

\[
\sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} \left[ (-1 + i \frac{3}{n})^2 + (-1 + i \frac{3}{n}) \right] \frac{3}{n}
\]

Let's expand and simplify:

\[
= \sum_{i=1}^{n} \left[ 1 - i \frac{3}{n} + i^2 \frac{9}{n^2} - 1 + i \frac{3}{n} \right] \frac{3}{n}
\]

\[
= \sum_{i=1}^{n} \left[ - i \frac{3}{n} + i^2 \frac{9}{n^2} \right] \frac{3}{n} = - \frac{9}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \sum_{i=1}^{n} i^2
\]

\[
= - \frac{9}{n^2} \left( \frac{n(n+1)}{2} \right) + \frac{27}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right)
\]

\[
= - \frac{9}{2} \left( \frac{n}{n} \left( \frac{n}{n} + \frac{1}{n} \right) \right) + \frac{27}{6} \left( \frac{n}{n} \left( \frac{n}{n} + \frac{1}{n} \right) \left( \frac{2n}{n} + \frac{1}{n} \right) \right)
\]

\[
= - \frac{9}{2} \left( 1(1+\frac{1}{n}) \right) + \frac{27}{6} \left( 1(1+\frac{1}{n})(2+\frac{1}{n}) \right)
\]

The above definite integral \( \int_{-1}^{2} (x^2 + x) \, dx \) is the limit of it's Riemann sum as \( n \to \infty \).

This limit is easily seen to be

\[-\frac{9}{2} \left( 1(1+0) \right) + \frac{27}{6} \left( 1(1+0)(2+0) \right)\]

\[-\frac{9}{2} + \frac{27}{6} = -\frac{27}{6} + \frac{27}{6} = \frac{9}{2}\]
Let's evaluate the same definite integral using the FTC2 which says
\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) = F(x) \bigg|_{a}^{b} \]
where \( \frac{dF}{dx} = f(x) \)

So:
\[ \int_{-1}^{2} (x^2 + x) \, dx = \left( \frac{x^3}{3} + \frac{x^2}{2} \right) \bigg|_{-1}^{2} \]
\[ = \frac{2^3}{3} + \frac{2^2}{2} - \left( \frac{(-1)^3}{3} + \frac{(-1)^2}{2} \right) \]
\[ = \frac{8}{3} + \frac{4}{2} - \left( -\frac{1}{3} + \frac{1}{2} \right) \]
\[ = \frac{8}{3} + \frac{4}{2} + \frac{1}{3} - \frac{1}{2} \]
\[ = \frac{16 + 12 + 2 - 3}{6} = \frac{27}{6} = \frac{9}{2} \]

Student: Write the Riemann Sum for
\[ \int_{1}^{3} x \, dx, \text{ then take the limit as } n \to \infty \]
of the Riemann Sum to evaluate the integral

Answer: \( a = 1, \ b = 3, \ \Delta x = \frac{2}{n}, \ x_i = 1 + i \frac{2}{n} \)

Riemann Sum = \( \sum_{i=1}^{n} \left( 1 + i \frac{2}{n} \right) \frac{2}{n} \)

Using formulas from first page
Simplifies to \( 2 \frac{2}{2} (1 + \frac{1}{n}) \)

The \( \lim_{n \to \infty} \sum_{i=1}^{n} \left( 1 + i \frac{2}{n} \right) \frac{2}{n} = \lim_{n \to \infty} \left( 2 + 2 \left( 1 + \frac{1}{n} \right) \right) = 4 \)