Eigenvalue Examples

Find eigenvalues, eigenvectors and eigenspaces for

\[ A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \]

Form \( A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & -2 - \lambda \end{bmatrix} \)

Characteristic equation is \( \det(A - \lambda I) = 0 \)

\[ \rightarrow (\lambda - 2)(-2 - \lambda) - 4 = 0 \]

\[ \rightarrow -2\lambda + 2\lambda + \lambda^2 - 4 = 0 \]

\[ \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0 \]

\( \lambda_1 = -3, \lambda_2 = 2 \) (Order is arbitrary)

For \( \lambda_1 = -3 \)

Form \( \begin{bmatrix} A - \lambda_1 I \end{bmatrix} \mathbf{u}_1 = 0 \)

\[ \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Eigenvector

\[ \begin{bmatrix} 4 \ 2 \\ 2 \ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \ 0 \ 0 \ 0 \end{bmatrix} \]

\( \mathbf{v}_1 = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \)

For \( \lambda_1 = -3 \), \( \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \), Eigenspace for \( \lambda_1 = -3 \) \( E_{\lambda_1 = -3} = \text{Span} \{ \mathbf{v}_1 \} \)

By convention, fractions are usually eliminated from eigenvectors.

For \( \lambda_2 = 2 \)

Form \( \begin{bmatrix} A - \lambda_2 I \end{bmatrix} \mathbf{u}_2 = 0 \)

\[ \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

For \( A \) matrices that are \( 2 \times 2 \) only, we can often take a short cut and not reduce to RREF.

Here the first row reads \(-1 \mathbf{v}_1 + 2 \mathbf{v}_2 = 0\)

and the second row reads \(2 \mathbf{v}_1 - 4 \mathbf{v}_2 = 0\)
These say the same thing. This happens because there must be a free variable column.

Here we can use either row, say \(-v_1 + 2v_2 = 0\).

This requires \(v_1 = 2v_2\).

Arbitrarily let \(v_2 = 1\) and then \(v_1 = 2\)

So thus \(\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\)

Thus \(\lambda_2 = 2\), \(\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\), \(E_{\lambda_2 = 2} = \text{Span} \{ \vec{v}_2 \}\)

If zeros are present, this may be confusing.

Then revert to doing RREF and write the solution in parametric format. The eigenvector(s) are the vectors which multiply the parameter(s).

You must use the RREF approach for matrices larger than 2x2 for easy solution.

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Student: Solve the eigenproblem for \(A = \begin{bmatrix} 3 & 0 \\ 5 & 2 \end{bmatrix}\)

Answer: \(\lambda_1 = 3\), \(\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}\), \(E_{\lambda_1 = 3} = \text{Span} \{ \vec{v}_1 \}\)

\(\lambda_2 = 2\), \(\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), \(E_{\lambda_2 = 2} = \text{Span} \{ \vec{v}_2 \}\)

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Solve the eigenproblem for \(A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\)

Answer: \(\lambda_1 = 0\), \(\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\), \(E_{\lambda_1 = 0} = \text{Span} \{ \vec{v}_1 \}\)

\(\lambda_2 = 2\), \(\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\), \(E_{\lambda_2 = 2} = \text{Span} \{ \vec{v}_2 \}\)
Example: Solve the eigenproblem for \( A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \)

Form \( A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \)

Characteristic equation is \( \det(A - \lambda I) = 0 \)

Expanding by the first column we obtain

\((1-\lambda) \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} + (1)(-1) \det \begin{bmatrix} 1 & -2 \\ 1 & -1-\lambda \end{bmatrix} = 0 \)

\((1-\lambda)( 2-\lambda)(-1-\lambda) - 1 + (-1-\lambda + 2) = 0 \)

\((1-\lambda)( -3 - \lambda + \lambda^2 ) + (1-\lambda) = 0 \)

\((1-\lambda)( \lambda^2 - \lambda - 2) = 0 \)

\((1-\lambda)( \lambda - 2)( \lambda + 1) = 0 \)

Notice that sometimes not rushing to multiply everything out allows easy identification of a factor.

So \( \lambda_1 = -1 \), \( \lambda_2 = 1 \), \( \lambda_3 = 2 \)

For \( \lambda_1 = -1 \), Form \( [A - \lambda_1 I] \mathbf{v}_1 = 0 \)

\( \begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

Eigenvector

\( \begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

and \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \)

\( \lambda_1 = -1 \), \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \), \( \mathbb{E}_{\lambda_1 = -1} = \text{Span} \{ \mathbf{v}_1 \} \)

Student: Continue for \( \lambda_2 = 1 \)

Answer: \( \lambda_2 = 1 \), \( \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \), \( \mathbb{E}_{\lambda_2 = 1} = \text{Span} \{ \mathbf{v}_2 \} \)
Student: Continue for \( \lambda_3 = 2 \)

Answer: \( \lambda_3 = 2, \ x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ E_{\lambda_3 = 2} = \text{Span} \left\{ x_3 \right\} \)

Student: For \( \mathbf{A} = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix} \)

Given that eigenvalues are \( \lambda_1 = 1, \ \lambda_2 = \lambda_3 = -3 \)

Find all eigenvectors and eigenspaces.

Answer: \( \lambda_1 = 1, \ \bar{x}_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \ E_{\lambda_1 = 1} = \text{Span} \left\{ \bar{x}_1 \right\} \)

\( \lambda_2 = \lambda_3 = -3, \ \bar{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \ \bar{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \)

\( E_{\lambda_2 = \lambda_3 = -3} = \text{Span} \left\{ \bar{x}_2, \bar{x}_3 \right\} \)

This Eigenspace of Dimension 2

For the eigenvalue -3 we say it has multiplicity 2

Since it is a factor \((\lambda + 3)^2\) of the characteristic equation. It happens in this problem to give two linear independent eigenvectors, \( \bar{x}_2 \) and \( \bar{x}_3 \). These constitute a basis set for a two dimensional eigenspace. Other set of vectors chosen from this vector space may also be used as basis sets.

As an example, the vector \( \bar{x} = -2 \bar{x}_2 + \bar{x}_3 \)

\( = -2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} \) is an eigenvector,

Since \( \mathbf{A} \bar{x} = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix} = -3 \bar{x} \)