Column, Row, and Null Space

Definitions

Span — The span of a set \( \{ \overline{v}_1, \ldots, \overline{v}_n \} \) of vectors in a vector space \( V \), denoted \( \text{Span}\{ \overline{v}_1, \ldots, \overline{v}_n \} \), is the set of all linear combinations of these vectors.

\[ \text{ie all } \overline{v} = c_1 \overline{v}_1 + c_2 \overline{v}_2 + \cdots + c_n \overline{v}_n \]

Column Space — For any \( m \times n \) matrix \( A \), the column space, denoted \( \text{Col}\, A \), is the span of the column vectors of \( A \), and is a subspace of \( \mathbb{R}^m \).

(Note: \( \mathbb{R}^m \) is formally a subspace of itself.)

Row Space — For any \( m \times n \) matrix \( A \), the row space, denoted \( \text{Row}\, A \), is the span of the row vectors of \( A \), and is a subspace of \( \mathbb{R}^n \).

Null Space — For any \( m \times n \) matrix \( A \), the null space, denoted \( \text{Null}\, A \) or sometimes \( \text{Ker}\, A \) (Kernel), is the subspace of \( \mathbb{R}^n \) consisting of all vectors \( \overline{x} \in \mathbb{R}^n \) for which \( A\overline{x} = \overline{0} \).
Basis of a Vector Space - The set \( \{ \vec{v}_1, \ldots, \vec{v}_n \} \) is a basis of the vector space \( V \) provided that:
- \( \text{Span} \{ \vec{v}_1, \ldots, \vec{v}_n \} = V \)
- \( \{ \vec{v}_1, \ldots, \vec{v}_n \} \) is linearly independent

Dimension of a Vector Space - The dimension of a vector space (or vector subspace) is the number of vectors in a basis set for that vector space (or vector subspace).

Rank of a Matrix - The rank of a matrix equals the number of columns with a pivot.

Nullity of a Matrix - The nullity of a matrix equals the number of columns without a pivot.

Thus for an \( m \times n \) matrix, \( A \):
\[
\text{Rank } A + \text{Nullity } A = n
\]

How to find basis for the Col \( A \), Row \( A \) and Null \( A \).

Starting with a \( m \times n \) matrix \( A \), use elementary row operations to obtain \( R \), the RREF of \( A \). Identify the number and position of the pivot columns of \( R \).
Rank $A = \text{number of pivot columns.}$  
Nullity $A = \text{number of columns without a pivot.}$

A basis for $\text{Row } A$ is the set of vectors consisting of all the non-zero rows in $R$:  
$\text{Row } A \subseteq \mathbb{R}^n$, $\text{Dim (Row } A) = \text{Rank } A$

A basis for $\text{Col } A$ is the set of columns in the original $A$ matrix corresponding in location to the column locations of pivots in $R$:  
$\text{Col } A \subseteq \mathbb{R}^m$, $\text{Dim (Col } A) = \text{Rank } A$

To find a basis for $\text{Null } A$, write the solution to $A\bar{x} = \bar{0}$ (using the RREF matrix $R$) in the standard format. It will look like  
$$\bar{x} = \bar{0} + \bar{w}_1 r + \bar{w}_2 s + ...$$
continuing for as many free variables needed. The column vectors $\bar{w}_i$ will be filled in with their components. The number of these vectors is Nullity $A$ and $\text{Nullity } A = n - \text{Rank } A$

A basis set for $\text{Null } A$ is the set  
$$\{ \bar{w}_1, \bar{w}_2, ..., \bar{w}_{\text{Nullity } A} \}$$

$\text{Null } A \subseteq \mathbb{R}^n$, $\text{Dim (Null } A) = \text{Nullity } A$
Example: \[ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \text{RREF} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Rank \( A = 2 \)

Nullity = 1

Basis for Row \( A = \{ (1,0,1), (0,1,1) \} \)

(Note: \((4,1,5) = 4(1,0,1) + 1(0,1,1)\))

Basis for Col \( A = \{ \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \} \)

(Note: \( \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} \))

Solution of \( A\bar{x} = \bar{0} \) is \( \bar{x} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} r \)

Only one column vector \( \bar{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) (Nullity = 1)

Basis for Null \( A = \{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \} \)

Check \( A \begin{bmatrix} -r \\ r \end{bmatrix} = \begin{bmatrix} -2r-r+2r \\ -2r+r+r \\ -4r-r+5r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

Do the same for:

- \( A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \text{RREF} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

- \( A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 7 & 10 \\ 4 & 2 & 4 & 6 \end{bmatrix} \rightarrow \text{RREF} \quad R = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)

- \( A = \begin{bmatrix} 0 & 0 & 2 & 2 & -2 \\ 2 & 2 & 6 & 14 & 4 \end{bmatrix} \rightarrow \text{RREF} \quad R = \begin{bmatrix} 1 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \)