Extra Credit Problem 6.94

An airplane in flight is subject to an air resistance force proportional to the square of its speed \( v \). But there is an additional resistive force because the airplane has wings. Air flowing over the wings is pushed down and slightly forward, so from Newton's third law the air exerts a force on the wings and airplane that is up and slightly backward (Fig. 6.30). The upward force is the lift force that keeps the airplane aloft, and the backward force is called induced drag. At flying speeds, induced drag is inversely proportional to \( v^3 \), so that the total air resistance force can be expressed by \( F_{\text{air}} = \alpha v^2 + \beta / v^2 \), where \( \alpha \) and \( \beta \) are positive constants that depend on the shape and size of the airplane and the density of the air. For a Cessna 150, a small single-engine airplane, \( \alpha = 0.30 \text{ N} \cdot \text{s}^2 / \text{m}^2 \) and \( \beta = 3.5 \times 10^4 \text{ N} \cdot \text{m}^2 / \text{s}^3 \). In steady flight, the engine must provide a forward force that exactly balances the air resistance force. a) Calculate the speed (in km/h) at which this airplane will have the maximum range (that is, travel the greatest distance) for a given quantity of fuel. b) Calculate the speed (in km/h) for which the airplane will have the maximum endurance (that is, will remain in the air the longest time).

![Figure 6.30 Challenge Problem](image)

Given that total air drag has equation

\[
F_{\text{drag}} = \alpha v^2 + \beta / v^2 \quad \text{for given } \alpha \text{ and } \beta.
\]

For a fixed amount of fuel we wish to maximize first the range and then the time in the air.

Let \( E \) be the energy content of a full load of fuel. Then

\[
E = P \times \Delta T, \quad P \text{ is power and } \Delta T \text{ the total time that the constant power level is maintained.}
\]

The engine and propeller make a forward force \( F \) to offset \( F_{\text{drag}} \). That is

\[
F - F_{\text{drag}} = 0 \quad \Rightarrow \quad F = F_{\text{drag}}
\]

Thus

\[
F = \alpha v^2 + \beta / v^2
\]
Power = \[ P = F \cdot v = (\alpha v^2 + \frac{\beta}{v^2}) \cdot v = \alpha v^3 + \frac{\beta}{v} \]

Maximize Range \( R \)

\[ R = v - \Delta T, \text{ but } \Delta T = \frac{E}{P} \]

So \( R = \frac{vE}{P} = \frac{vE}{\alpha v^3 + \frac{\beta}{v}} = \frac{v^2 E}{\alpha v^4 + \beta} \)

\[ \frac{dR}{dv} = (\alpha v^4 + \beta) \left( 2vE \right) - v^2 E \left( 4 \alpha v^3 \right) \]

\[ \text{Simplify} \quad v^4 = \frac{\beta}{\alpha}, \quad v = \left( \frac{\beta}{\alpha} \right)^{\frac{1}{4}} \]

Maximize duration, i.e. \( \Delta T \)

\[ \Delta T = \frac{E}{P} = \frac{vE}{\alpha v^4 + \beta} \]

\[ \frac{d\Delta T}{dv} = 0 \Rightarrow \quad v = \left( \frac{\beta}{3\alpha} \right)^{\frac{1}{4}} \]