Span, Linear Independence and Basis for Polynomials

The properties of spanning, linear independence and identifying basis sets for polynomials in vector space \( \mathbb{P}_n \) can be investigated in terms of the same properties of related vectors in \( \mathbb{R}^{n+1} \) by use of an isomorphism between \( \mathbb{P}_n \) and \( \mathbb{R}^{n+1} \).

An isomorphism is an invertible mapping which preserves essential information. For example, between \( \mathbb{P}_2 \) and \( \mathbb{R}^3 \) the usual isomorphism mapping is:

\[
\begin{bmatrix}
a_0 \\
a_1 \\
a_2
\end{bmatrix} \leftrightarrow a_0 + a_1 t + a_2 t^2
\]

Let's investigate the set

\[
S = \{ -t + t^2, 1 + t, 2 + t^2 \}
\]

of polynomials from \( \mathbb{P}_2 \).

\[
-t + t^2 \leftrightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]

\[
1 + t \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\] and

\[
2 + t^2 \leftrightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\]

Thus we will analyse the set \( S' = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\} \)

of vectors from \( \mathbb{R}^3 \).

We already know how to do this:

Enter the vectors as columns in a matrix \( A \)

\[
A = \begin{bmatrix}
0 & 1 & 2 \\
-1 & 0 & 1
\end{bmatrix}
\]
Then do EROS to reach an echelon form:

\[
\begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 1 \\
0 & 1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
-1 & 1 & 0 \\
0 & 1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[R_1 \leftrightarrow R_2 \quad R_2' = R_2 + R_1 \quad R_3' = R_3 - R_2\]

I have circled the pivots in the echelon form.

- A pivot in every column ⇔ column vectors are linear independent
- A pivot in every row ⇔ column vectors span \( \mathbb{R}^3 \)
- Both linear independent and span ⇔ the set of column vectors may be used as a basis set for \( \mathbb{R}^3 \)

Because of the isomorphism, the same conclusions are true for the polynomials in \( S \).

I.e., The polynomials are linear independent, they span \( \mathbb{P}_2 \), and they constitute a basis set for \( \mathbb{P}_2 \).

Let's write the polynomial \( 3 - 2t - t^2 \) as a linear combination of the polynomials in set \( S \). I.e we want to find \( x_1, x_2, x_3 \) which give

\[3 - 2t - t^2 = x_1(-t + t^2) + x_2(1 + t) + x_3(2 + t^2)\]

We apply the isomorphism

\[3 - 2t - t^2 \leftrightarrow \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}\]

and the problem becomes

\[
\begin{bmatrix}
0 & 1 & 2 \\
-1 & 1 & 0 \\
1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} =
\begin{bmatrix}
3 \\
-2 \\
-1 \\
\end{bmatrix}
\]

Augment and RREF, we obtain

\[
\begin{bmatrix}
1 & 0 & 0 & -7 \\
0 & 1 & 0 & -9 \\
0 & 0 & 1 & 6 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} =
\begin{bmatrix}
-7 \\
-9 \\
6 \\
\end{bmatrix}
\]
Let's check:
\[-7(-t+t^2)-9(1+t)+6(2+t^2)\]
\[= 7t - 7t^2 - 9t + 12 + 6t^2\]
\[= (-9+12) + (7-9)t + (-7+6)t^2\]
\[= 3 - 2t - t^2 \quad \text{yes}\]

Student: Check spanning, linear independence and determine if a basis set for:

- \(S = \{-t+t^2, 1+t, 2+t^2, -1+t\}\)

**Answer:** Obtain \(A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}\)

EROS to echelon → \(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}\)

\(S\) is not linear independent but does span \(\mathbb{P}_2\). Not a basis set. Echelon forms are not unique, but you will see a pivot in every row but not in every column for all correct echelon forms.

- \(S = \{1-2t+t^2, -1+t\}\)

**Answer:** Obtain \(A = \begin{bmatrix} 1 & -1 \\ -2 & 1 & 0 \end{bmatrix}\)

EROS to echelon → \(\begin{bmatrix} 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\)

\(S\) is linear independent, but does not span \(\mathbb{P}_2\). Not a basis set.
\[ S = \{ 2t + t^2, -t + t^2, 1 + t^2 \} \]

Answer: Obtain \( A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \)

EROS to echelon \( \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \)

\( S \) is not linear independent and does not span \( \mathbb{P}_2 \). Not a basis set.