Matrix Inverse

The algebra of scalars gives:
Given \( ax = b \)
\[ a^{-1}ax = a^{-1}b \quad a \neq 0 \]
\[ 1x = a^{-1}b \]
\[ x = a^{-1}b = \frac{b}{a} \]

There is no division by a matrix, but there is a matrix inverse, denoted as \( A^{-1} \). A must be a square matrix, and even then, a given matrix \( A \) may not have an inverse.

Given \( A\bar{x} = \bar{b} \)
\[ A^{-1}A\bar{x} = A^{-1}\bar{b} \]
\[ I\bar{x} = A^{-1}\bar{b} \quad \text{This only works} \]
\[ \bar{x} = A^{-1}\bar{b} \quad \text{when} \bar{x} \text{has a unique solution.} \]

The \(-1\) is not a power, ie. \( A^{-1} \neq \frac{1}{A} \), it is just naming that \( A^{-1} \) is the inverse of \( A \).

Note that \( I \) is an \( n \times n \) matrix consisting of 1's down the diagonal and 0's elsewhere.
\[ I\bar{v} = \bar{v}I = \bar{v} \quad \text{and} \quad IA = AI = A \]

Remember that in general, \( AB \neq BA \).

Some properties of the matrix inverse are:
\[ A^{-1}A = AA^{-1} = I \]
\[ (A^{-1})^{-1} = A \]
\[ (AB)^{-1} = B^{-1}A^{-1} \]
\[ (A^T)^{-1} = (A^{-1})^T \]
In all cases, \( A \) and \( B \) assumed to be invertible.
How to find $A^{-1}$ given $A$?

- First, $A$ must be a square matrix, i.e., $n \times n$.
- Even then, $A^{-1}$ does not exist for all $A$'s.
- Our preferred process when $n \geq 3$ is:
  Augment $A$ with the $n \times n$ identity matrix.
  Using EROS, reduce $A$ to RREF.
  \[ [A; I] \xrightarrow{\text{RREF}} [I; A^{-1}] \]
  On the right after RREF, if $A$ has changed to $I$, then $A^{-1}$ is in the augmenting location. If the RREF form of $A$ is not $I$, then $A^{-1}$ does not exist.

- When $A$ is of dimension $2 \times 2$, this process still works, however it is easy to memorize the answer. For
  \[
  A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \begin{bmatrix} d-b & -b \\ \frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}
  \]
  Note: this inverse does not exist if $ad-bc = 0$.

This term, $ad-bc$, is called the determinant of the given $2 \times 2$ matrix $A$.
The determinant generalizes and is a unique number for any $n \times n$ matrix, and in general, $A^{-1}$ exists $\iff$ $\det A \neq 0$.

Finding the determinant, $\det A$, will be our next topic.
Example: Given a 2x2 matrix, $A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$, find $A^{-1}$.

$A^{-1} = \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -3/2 & 1 \\ -1 & 1/2 \end{bmatrix}$

Check: $AA^{-1} = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} -3/2 & 1 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For Student: Given $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$, find $A^{-1}$ and check.

Given $A = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$, find $A^{-1}$

Answers: For first, $A^{-1} = \begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}$ For second, $A^{-1}$ does not exist.

Example: Given a 3x3 matrix, $A = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, find $A^{-1}$.

Form $\begin{bmatrix} 1 & -3 & 0 & 1 & 0 & 0 \\ 2 & -5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ Do EROS to reach RREF

$R_2^* = R_2 - 2R_1$ $\begin{bmatrix} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$

$R_1^* = R_1 + 3R_2$ $\begin{bmatrix} 1 & 0 & 0 & 5 & 3 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix}$

Successful, so $A^{-1} = \begin{bmatrix} 5 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

Check: $AA^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For Student: Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, find $A^{-1}$ and check.

Answer: $A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$
If we have $A\vec{x} = \vec{b}$ and a unique solution then the answer for $\vec{x}$ is

$$\vec{x} = A^{-1}\vec{b}$$

But why would we do the solution this way instead of

$$[A;\vec{b}] \xrightarrow{\text{RREF}}$$

This is easier since we have only one augmenting column.

Finding $A^{-1}$ is useful if there is a single matrix $A$ but many different vectors $\vec{b}$. Then once $A^{-1}$ is found it can be used repeatedly with the different $\vec{b}$ vectors by simple matrix-vector multiplication.

Also, there are numerous more advanced uses of matrices in which an inverse of a matrix is needed. One example is optimal control theory.