Matrix Inverse

The algebra of scalars gives:
Given \( ax = b \)
\[ a^{-1}ax = a^{-1}b, \quad a \neq 0 \]
\[ 1x = a^{-1}b \]
\[ x = a^{-1}b = \frac{b}{a} \]

There is no division by a matrix, but there is a matrix inverse, denoted as \( A^{-1} \). A must be a square matrix, and even then, a given matrix \( A \) may not have an inverse.

Given \( A\bar{x} = \bar{b} \)
\[ A^{-1}A\bar{x} = A^{-1}\bar{b} \]
\[ I\bar{x} = A^{-1}\bar{b} \quad \text{This only works} \]
\[ \bar{x} = A^{-1}\bar{b} \quad \text{when } \bar{x} \text{ has a unique solution.} \]

The \(-1\) is not a power, i.e. \( A^{-1} \neq \frac{1}{A} \), it is just naming that \( A^{-1} \) is the inverse of \( A \).

Note that \( I \) is an \( n \times n \) matrix consisting of \( 1 \)'s down the diagonal and \( 0 \)'s elsewhere.
\[ I\bar{x} = \bar{y}I = \bar{y} \quad \text{and} \quad IA = AI = A \]

Remember that in general \( AB \neq BA \).

Some properties of the matrix inverse are:
\[ A^{-1}A = AA^{-1} = I \]
\[ (A^{-1})^{-1} = A \quad \text{In all cases, } A \text{ and} \]
\[ (AB)^{-1} = B^{-1}A^{-1} \quad B \text{ assumed to be} \]
\[ (A^T)^{-1} = (A^{-1})^T \quad \text{invertible.} \]
How to find $A^{-1}$ given $A$?

- First, $A$ must be a square matrix, i.e. $n \times n$.
- Even then, $A^{-1}$ does not exist for all $A$'s.
- Our preferred process when $n \geq 3$ is:
  
  Augment $A$ with the $n \times n$ identity matrix. Using EROS, reduce $A$ to RREF.
  
  $[A: I] \xrightarrow{\text{RREF}} [I: A^{-1}]$
  
  On the right after RREF, if $A$ has changed to $I$, then $A^{-1}$ is in the augmenting location. If the RREF form of $A$ is not $I$, then $A^{-1}$ does not exist.

- When $A$ is of dimension $2 \times 2$, this process still works, however it is easy to memorize the answer. For

  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \begin{bmatrix} d-b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ \frac{d-c}{ad-bc} & \frac{ad-bc}{ad-bc} \\ -c & a \end{bmatrix}$

  Note: this inverse does not exist if $ad-bc = 0$.

  This term, $ad-bc$, is called the determinant of the given $2 \times 2$ matrix $A$. The determinant generalizes and is a unique number for any $n \times n$ matrix, and in general, $A^{-1}$ exists $\iff \det A \neq 0$.

Finding the determinant, $\det A$, will be our next topic.
Example: Given a 2x2 matrix, \( A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix} \), find \( A^{-1} \).

\[
A^{-1} = \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -3/2 & 1 \\ -1 & 1/2 \end{bmatrix}
\]

\(-12+16\)

Check: \( AA^{-1} = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} -3/2 & 1 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I\)

For Student: Given \( A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix} \), Find \( A^{-1} \) and check.

Given \( A = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} \), Find \( A^{-1} \)

Answers: For first, \( A^{-1} = \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix} \) For second, \( A^{-1} \) does not exist.

Example: Given a 3x3 matrix, \( A = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 1 & 1 \end{bmatrix} \), find \( A^{-1} \).

Form \( \begin{bmatrix} 1 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 1 & 1 \end{bmatrix} \) Do EROS to reach RREF

\[
R_2^* = R_2 - 2R_1 \\
\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}
\]

\[
R_1^* = R_1 + 3R_2 \\
\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}
\]

\[
R_3^* = R_3 + R_2 \\
\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}
\]

Successful, so \( A^{-1} = \begin{bmatrix} -5 & 3 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \)

Check: \( AA^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 3 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

For Student: Given \( A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \), Find \( A^{-1} \) and check.

Answer: \( A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \)
If we have $A\bar{x} = \bar{b}$ and a unique solution then the answer for $\bar{x}$ is

$$\bar{x} = A^{-1}\bar{b}$$

But why would we do the solution this way instead of

$$[A; \bar{b}] \xrightarrow{\text{RREF}} ?$$

This is easier since we have only one augmenting column.

Finding $A^{-1}$ is useful if there is a single matrix $A$ but many different vectors $\bar{b}$. Then once $A^{-1}$ is found it can be used repeatedly with the different $\bar{b}$ vectors by simple matrix-vector multiplication.

Also, there are numerous more advanced uses of matrices in which an inverse of a matrix is needed. One example is optimal control theory.