Work and Energy

A force acting on a moving object may do work on the object. The amount of work depends on the component of force in the direction of displacement and the amount of displacement.

In general, the work is a path integral of a vector field (Math6A)

\[
\text{Work} = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{F}(t)) \cdot d\mathbf{F}, \quad \text{Nm} \in \text{J (Joule)}
\]

\( \mathbf{F}(t) = (x(t), y(t), z(t)) \) is a parameterization of the path, and the \( \cdot \) indicates a dot product.

When \( \mathbf{F} \) is of constant magnitude \( F \) and the path is a straight displacement \( d \), the work is simply:

\[
\text{Work} = F \cdot d \cos \theta
\]

For example, a mass displacing down a distance \( d \) has work done on it by the force of gravity:

\[
\int d \mathbf{F}_g = mg \quad \text{Work} = F_g d \cos 0^\circ = mgd
\]

A mass displacing a distance \( d \), as a kinetic friction force acts on it, has work done on it:

\[
\int d \mathbf{F}_k = -F_k \quad \text{Work} = F_k d \cos 180^\circ = -F_k d \quad \text{(Note negative)}
\]
A normal force does no work if the object moves only perpendicular to the force:

\[ \text{Work} = nd \cos 90^\circ = nd \times 0 = 0 \]

However, a ball colliding with a wall has work done on it by the normal force of the wall as the ball makes a small displacement during the short time of contact:

\[ \text{Work} = nd \cos 180^\circ = -nd \]

The kinetic energy of a point mass with velocity \( \mathbf{v} \) is:

\[ K = \frac{1}{2} m v^2, \quad k \frac{m^2}{s^2} = J \]

\( v \) is the speed \( v = \sqrt{v_x^2 + v_y^2 + v_z^2} \)

The sum of all work done on a point mass as it moves from some initial condition \( i \), to a final condition \( f \), is

\[ \text{Work Total} = \Delta K = K_f - K_i \]

There are two categories of forces, conservative and non-conservative. Conservative forces do zero net work as
an object follows a closed path and returns to its initial position. Also, the work done does not depend on the details of the path, but only on the initial and final locations. The work done by a conservative force is equal to the negative of the change of a potential energy function:

\[ \text{Work}_{\text{con}} = -\Delta U = -(U_f - U_i) \]

Our present examples of conservative forces are gravity force and spring force.

For gravity:

\[ U_g = mg \Delta y \]

The location of \( y = 0 \) is arbitrary and usually is chosen for convenience of calculation.

For ideal springs, the force is

\[ F_{sp} = -k \Delta x \]

where \( k \) is the spring coefficient with units \( \text{N/m} \) and \( \Delta x \) is the elongation (+) or compression (-) of the spring from its natural condition. The potential energy of such an ideal spring is:

\[ U_{sp} = \frac{1}{2} k \Delta x^2 \]

Other conservative forces and their potential energy functions will be introduced later.
If we use potential energy functions for gravity and springs, we may change Eqn 1 to:

\[ \text{Work}_{N.c.} - \Delta U_g - \Delta U_{sp} = \Delta K \]

or

\[ \text{Work}_{N.c.} = \Delta K + \Delta U_g + \Delta U_{sp} = \Delta E = E_f - E_i \]

where

\[ E = K + U_g + U_{sp} \]

For the present, non-conservative work, \( \text{Work}_{N.c.} \) is work done by friction, external applied forces (pushes, pulls, tensions) and normal forces.