Elliptical Orbit Problem

12.75 Consider a spacecraft in an elliptical orbit around the earth. At the low point, or perigee of its orbit, it is 400 km above the earth’s surface; at the high point, or apogee, it is 4000 km above the earth’s surface. a) What is the period of the spacecraft’s orbit? b) Using conservation of angular momentum, find the ratio of the spacecraft’s speed at perigee to its speed at apogee. c) Using conservation of energy, find the speed at perigee and the speed at apogee. d) It is necessary to have the spacecraft escape from the earth completely. If the spacecraft’s rockets are fired at perigee, by how much would the speed have to be increased to achieve this? What if the rockets were fired at apogee? Which point in the orbit is more efficient to use?

\[ R_p = R_e + h_p = 6.38 \times 10^6\ m + 400\ km = 6.78 \times 10^6\ m \]
\[ R_a = R_e + h_a = 6.38 \times 10^6\ m + 4000\ km = 10.38 \times 10^6\ m \]
\[ a = \frac{R_p + R_a}{2} = 8.58 \times 10^6\ m \]

\[ P_A: \text{Find Period} \quad T = \frac{2\pi a^{3/2}}{\sqrt{GM}} \]
\[ M = M_e = 5.97 \times 10^{24}\ kg \]
\[ G = 6.673 \times 10^{-11}\ Nm^2/kg^2 \]
\[ T = 7.91 \times 10^3\ s \quad (2.20\ hr.) \]

PB: Relate speed at perigee to speed at apogee. Angular momentum, \( L \), is conserved because no torque acts on the system.

Notice, \( u_a \perp R_a \) and \( u_p \perp R_p \)

\[ L = mu_a R_a = mu_p R_p \]
\[ \text{Thus} \quad \frac{u_p}{u_a} = \frac{R_a}{R_p} = 1.531 \]

PC: Energy is conserved because Work Non-conservative is zero.
Thus \( K_a + U_a = K_p + U_p \)
\( K_p - K_a = U_a - U_p \)
\[
\frac{1}{2} m v_p^2 - \frac{1}{2} m v_a^2 = -\frac{GMm}{R_a} - \left( -\frac{GMm}{R_p} \right)
\]
\( v_p^2 - v_a^2 = 2GM \left( \frac{1}{R_p} - \frac{1}{R_a} \right) \)
But already obtained (PB) that
\( v_p = 1.531 v_a \)
Thus \((1.531^2 - 1) v_a^2 = 2GM \left( \frac{1}{R_p} - \frac{1}{R_a} \right)\)
Solve for \( v_a = 5.51 \times 10^3 \text{ m/s} \)
and so \( v_p = 8.43 \times 10^3 \text{ m/s} \)

**PD:** To escape Earth, need a total energy of zero \( (E_{\text{final}} = 0 \text{ because } U \rightarrow 0 \text{ as } r \rightarrow \infty \text{ and } K \rightarrow 0 \text{ as } v = 0 \text{ at } r \rightarrow \infty) \)
So Need
\( K_p + U_p = 0 \)
Looking for new velocity at perigee
\[
\frac{1}{2} m v_{p,\text{escape}}^2 = \frac{GMm}{R_p}
\]
\( v_{p,\text{escape}} = \sqrt{\frac{2GM}{R_p}} = 1.084 \times 10^4 \text{ m/s} \)
An increase of \( v_{p,\text{escape}} - v_p = 2.41 \times 10^3 \text{ m/s} \)
Similar calculation at apogee gives
\( v_{a,\text{escape}} = 8.761 \times 10^3 \text{ m/s} \)
For an increase needed of \( 3.25 \times 10^3 \text{ m/s} \)
**PE:** Most efficient to change velocity at perigee