Transformation

A transformation (or mapping), denoted \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a rule that assigns each vector \( \vec{x} \) in \( \mathbb{R}^n \) (the domain) an output vector \( T(\vec{x}) \) in \( \mathbb{R}^m \) (the codomain). \( T(\vec{x}) \) is often called the image of \( \vec{x} \) and the set of all possible images is called the range of \( T \).

A transformation is a linear transformation if for \( \vec{u}, \vec{v} \in \mathbb{R}^n \) and \( c \) any scalar,
\[
T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})
\]
and
\[
T(c\vec{u}) = cT(\vec{u}).
\]
Notice that \( \vec{u} + \vec{v} \) and \( c\vec{u} \) occur in \( \mathbb{R}^n \), while \( T(\vec{u}) + T(\vec{v}) \) and \( cT(\vec{u}) \) occur in \( \mathbb{R}^m \).

Example: Is \( T(\begin{bmatrix} x \\ y \end{bmatrix}) = xy \) a linear transformation (LT)?

Let \( \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \), then \( c\vec{u} = \begin{bmatrix} cx \\ cy \end{bmatrix} \)
\[
T(c\vec{u}) = (cx)(cy) = c^2 xy
\]
and \( cT(\vec{u}) = c(xy) = cxy \) So NO

Students try: Is \( T(\begin{bmatrix} x \\ y \end{bmatrix}) = x + 2y + a, a \in \mathbb{R}, aLT? \)

Answer NO

The only value of a allowed is \( a = 0 \).

Notice: \( T(c\vec{u}) = cT(\vec{u}) \Rightarrow T(\vec{0}) = \vec{0} \)
A very important result is that every L.T. \( T : \mathbb{R}^n \to \mathbb{R}^m \) is equivalent to a matrix multiplication, i.e. \( T(\vec{x}) = A\vec{x} \) where \( A \) is an \( m \times n \) matrix called the standard matrix for the L.T. This is often denoted as: \( \vec{x} \mapsto A\vec{x} \).

Now we see why we consider \( \mathbb{R}^n \) as the input space and \( \mathbb{R}^m \) as the output space — so that \( A \) is of dimension \( m \times n \).

This matrix is easy to compute since it is:

\[
A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}
\]

i.e., the \( j \)th column of \( A \) is just the transformation applied to the \( j \)th standard basis vector \( \vec{e}_j \). Remember \( \vec{e}_j \in \mathbb{R}^n \) consists of a 1 at the \( j \)th location and all other entries are 0.

Example: \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by the rule

\( T(\vec{x}) = T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, -x_1 + 4x_2) \)

Note, I have written vectors horizontally to save space. But remember, \( T(\vec{e}_j) \) is still the \( j \)th column of \( A \).

\( \vec{e}_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \), so that means the inputs are \( x_1 = 1, x_2 = 0, x_3 = 0 \).

Thus \( T(\vec{e}_1) = (2 \cdot 1 - 0 + 0, -1 + 4 \cdot 0) = (2, -1) \)

This becomes the first column of \( A \).
Continuing, for $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, \( T(\vec{e}_2) = (-1, 4) \) and for $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, \( T(\vec{e}_3) = (1, 0) \)

Thus \( A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & 0 \end{bmatrix} \)

Check \( A \vec{x} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ -x_1 + 4x_2 + 0x_3 \end{bmatrix} \)

the first output element is \( 2x_1 - x_2 + x_3 \) and the second output element is \( -x_1 + 4x_2 \).

Student: The effect of a combination of a certain horizontal shear and a certain vertical expansion is illustrated:

![Diagram of horizontal shear and vertical expansion](image)

For this \( T: \mathbb{R}^2 \to \mathbb{R}^2 \), find \( A \)

Answer: \( A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \)

A mapping \( T: \mathbb{R}^n \to \mathbb{R}^m \) is said to be onto \( \mathbb{R}^m \) if each \( \vec{b} \) in \( \mathbb{R}^m \) is the image of at least one \( \vec{x} \) in \( \mathbb{R}^n \).
A linear transformation is \textit{onto} if and only if the standard matrix $A$ has a pivot in every row.

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be \underline{one-to-one} if each $b$ in $\mathbb{R}^m$ is the image of at most one $x$ in $\mathbb{R}^n$.

$$
\begin{array}{c}
\text{T is one-to-one} \\
\text{T is not one-to-one}
\end{array}
$$

A linear transformation is \underline{one-to-one} if and only if the standard matrix $A$ has a pivot in every column.

\textbf{Student: Test the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ for which we found the matrix for the properties \underline{onto} and \underline{one-to-one}.}

\textbf{Answer:} $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & 0 \end{bmatrix}$

\text{RREF} $\begin{bmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{1}{7} \end{bmatrix}$

\text{Pivot in every row} \Rightarrow T \text{ is \underline{onto}}

\text{Note: Pivot in every column} \Rightarrow T \text{ is not one-to-one}